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Polynomial Approximation in $\mathbf{L_p(R, d\mu)}$. I

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For arbitrary $w : R \to [0,1]$ the general form of the continuous linear functionals on the space C_w^0 of all functions f continuous on the real line, $\lim_{|x| \to +\infty} w(x) f(x) = 0$, equipped with seminorm $||f||_w := \sup_{x \in R} w(x)|f(x)|$, is found. The weighted analog of the Weierstrass polynomial approximation theorem and a new version of M.G. Krein's theorem about partial fraction decomposition of the reciprocal of an entire function are established. New descriptions of the Hamburger and Krein classes of entire functions are obtained. Preprint includes the final representation of all those measures μ for which algebraic polynomials are dense in $L_p(R, d\mu)$.

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Polynomial Approximation in $\mathbf{L_p(R, d\mu)}$.
Part I. General Results and Representation Theorem.

Andrew G. Bakan

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CONTENTS

INTRODUCTION

This paper is devoted to the weighted polynomial approximation problem on the real line.

Let $w(x)$ be a nonnegative function of real values x, such that for each $n = 0, 1, 2, \ldots$, $x^n w(x)$ is bounded. In 1924 S.Bernstein [10] asked for conditions on w such that the algebraic polynomials $\mathcal{P}$ are dense in the space C^0_w of all functions f continuous on $\mathbb{R}$, satisfying $w(x)f(x) \to 0$ as $|x| \to +\infty$, where C^0_w is equipped with the seminorm $||f||_w := \sup\limits_{x\in\mathbb{R}} w(x)|f(x)|$ (for a more explicit survey see [1, 30, 32, 40, 41]).

In 1937 S. Isumi and T. Kawata [20] showed that if functions $w(x)$ and $-\log w(e^x)$ are even and convex on the real line, respectively, then algebraic polynomials $\mathcal{P}$ are dense in the space C^0_w if and only if

$$\int\limits_{\mathbb{R}} \frac{\log w(x)}{1+x^2}\, dx \quad = \quad -\infty\ . \tag{1}$$

In 1947 N. Akhiezer and S. Bernstein (see [32, 1]) proved that a necessary and sufficient condition for the density of $\mathcal{P}$ in C^0_w is that

$$\sup_{P\in\mathfrak{M}_w} \int\limits_{\mathbb{R}} \frac{\log|P(x)|}{1+x^2}\, dx \quad = \quad +\infty\ , \tag{2}$$

where $\mathfrak{M}_w := \{P \in \mathcal{P} \mid w(x)|P(x)| \le 1 + |x| \ \ \forall\, x \in \mathbb{R}\}$. It was shown in 1956 by S. Mergelyan [32] that condition (2) is equivalent to

$$\int\limits_{\mathbb{R}} \frac{\log\left[\sup\limits_{P\in\mathfrak{M}_w} |P(x)|\right]}{1+x^2}\, dx \quad = \quad +\infty\ . \tag{3}$$

In 1959 L. de Branges [12] obtained a remarkable theorem for functions w which are positive and continuous on the real line. He proved that $\mathcal{P}$ is dense in C^0_w if and only if for any real entire function F of exponential type all whose zeros Λ_F are real and simple and which satisfies:

$$\int\limits_{\mathbb{R}} \frac{\log^+|F(x)|}{1+x^2}\, dx \quad < \quad +\infty\ , \tag{4}$$

where $\log^+ x := \max\{0, \log x\}$, $x \ge 0$, the following relation holds:

$$\sum_{\lambda\in\Lambda_F} \frac{1}{w(\lambda)|F'(\lambda)|} \ = \ \infty. \tag{5}$$

In 1989 B.Ja.Levin [30] extended conditions (2) and (3) to all spaces $L_p(\mathbb{R}, d\mu)$, $1 \le p < \infty$, where μ is a positive Borel measure on the real line with finite moments of any order:

$$\int\limits_{\mathbb{R}} |x|^n\, d\mu(x) \ < \ \infty \ \ \forall\, n = 0, 1, 2, \ldots\ ,$$

and unbounded support. He proved that each of the conditions (2) and (3) represents a necessary and sufficient condition for polynomials to be dense in $L_p(\mathbb{R}, d\mu)$ where $\mathfrak{M}_w$ is replaced by

$$\mathfrak{M}_p \ := \ \left\{ P \in \mathcal{P} \mid \quad \int\limits_{\mathbb{R}} \frac{|P(x)|^p}{(1+|x|)^p} \, d\mu(x) \ \le \ 1 \right\} .$$

It should be noted here that the condition (3) for $p = 2$ coincides with M. Riesz's theorem (1922) in classical moment theory [35; 36; 2, Th. 2.4.1].

In 1996 M. Sodin and P. Yuditskii [41] found a simpler proof of de Branges theorem and proved its validity assuming only the upper semicontinuity w on $\mathbb{R}$. Moreover, in de Branges condition (5), they have replaced the function F by an arbitrary real entire function B of minimal exponential type with only simple real zeros $\Lambda_B \subseteq \{x \in \mathbb{R} \mid w(x) > 0\}$. In 1998 M. Sodin and A. Borichev [11] established a criterion similar to (5) for polynomial density in all spaces $L_p(\mathbb{R}, d\mu)$, $1 \le p < \infty$, under the condition that measure μ is discrete and for some positive number a :

$$\sum_{\lambda \in \ \mathrm{supp}\, \mu} \frac{1}{(1+|\lambda|)^a} \ < \ \infty .$$

In the first part of that paper we will extend de Branges condition (5) to all spaces $L_p(\mathbb{R}, d\mu)$, $1 \le p < \infty$, without any additional assumption about measure μ , and in the second part, obtain a new analytical proof of these conditions, showing their real nature from the point of view of extremal problems theory.

In the first Chapter for an arbitrary function $w : \mathbb{R} \to [0,1]$, we give a complete description of the Banach space B_w^0 associated with the seminormed space C_w^0 (Theorem 1.1). This description, under the condition that $\mathcal{P}$ is dense in C_w^0 , makes it possible in Theorem 1.2 to characterize all functions $f : \{ x \in \mathbb{R} \mid w(x) > 0 \} \to \mathbb{R}$ which can be approximated in the seminorm $||\cdot||_w$ by polynomials. That is why Theorem 1.2 represents a supplement to S. Mergelyan's theorem [32, Th.7] in those cases when polynomials are dense in the space C_w^0 . Besides that, the weighted analog of the Weierstrass polynomial approximation theorem is derived from Theorem 1.2 when the set $\{ x \in \mathbb{R} \mid w(x) > 0 \}$ is bounded.

In the Chapter II, Hamburger criterion of polynomial density, known in the classical theory of moments, has been extended to all spaces $L_p(\mathbb{R}, d\mu)$, $1 \le p < \infty$, and C_w^0 .

Chapter III contains a new version of M. Krein's theorem about the partial fraction decomposition of the reciprocal of an entire function (Theorem 3.1). Its Corollaries 3.1 and 3.2 give a new characterization of the Hamburger and Krein classes of entire functions. Strictly normal polynomial families are introduced in section 3.4 and sufficient conditions to have such property are found in the Theorem 3.3.

Chapter IV includes the main result of this paper (Theorem 4.1), which allows us to formulate conditions similar to (5) in all spaces $L_p(\mathbb{R}, d\mu)$, $1 \le p < \infty$.

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CHAPTER I. Banach space associated with seminormed space C_w^0

1.1. Notations. Everywhere below in this Chapter, only real linear spaces and spaces of real-valued functions are considered. It is worth to remind [13, 1.10.2] that the pair $X = (\mathcal{L}(X), \|\cdot\|_X)$ is called a seminormed space if $\mathcal{L}(X)$ is a linear space and $\|\cdot\|_X$ is defined on $\mathcal{L}(X)$ seminorm. We will write X instead of $\mathcal{L}(X)$, i.e. $X = (X, \|\cdot\|_X)$. Denote by X^* the Banach space [13, 1.10.6] of all linear continuous functionals L on the seminormed space X , equipped with norm $\|L\| := \sup\{|L(x)| \mid x \in X,\ \|x\|_X \leq 1\}$. For two seminormed spaces X and Y notation $X \equiv Y$ indicates that X and Y coincide identically, i.e. $X = Y$ and $\|x\|_X = \|x\|_Y \ \ \forall x \in X$. For two normed spaces X and Y notation $X \cong Y$ means that X and Y are *isometric*, i.e. there exists such linear transformation $U : X \to Y$, that: a) $U(X) = Y$; b) $\|U(x)\|_Y = \|x\|_X \ \ \forall x \in X$. If $X = (X, \|\cdot\|_X)$ is a seminormed space then the normed factor space $X \setminus N_X = \Big(X \setminus N_X,\ \|\cdot\|_{X\setminus N_X}\Big)$ whose elements are classes $\pi(x) := x + N_X$, $\|\pi(x)\|_{X\setminus N_X} := \|x\|_X \ \ \forall x \in X$ and $N_X := \{\, x \in X \mid \|x\|_X = 0\}$ is said to be [13, 1.10.2] a normed space *associated* with seminormed space X .

Let $A \subseteq \mathbb{R}$. The closure of A is denoted by $\overline{A}$, and $\chi_A(x) := \begin{cases} 1, & x \in A; \\ 0, & x \notin A. \end{cases}$ Let $C(A)$ denote the linear space of all continuous on A functions $f : A \to \mathbb{R}$; $C^0(A) := \Big(C^0(A),\ \|\cdot\|_{C(A)}\Big)$ - Banach space of such bounded on A , i.e. $\|f\|_{C(A)} := \sup\limits_{x\in A} |f(x)| < \infty$, functions $f \in C(A)$ that $\lim\limits_{x\in A,\ |x|\to\infty} f(x) = 0$, if A is unbounded; $\mathbb{Z}_0$ - the set of all nonnegative integers. Function $f \in C(\mathbb{R})$ is called *compactly supported* if it's equal to zero outside of some compact subset of the real line.

For $A \subseteq B \subseteq \mathbb{R}$ and $h : B \to \mathbb{R}$ symbol $h \restriction_A$ denotes the function $h \restriction_A : A \to \mathbb{R}$, $h \restriction_A (x) = h(x) \ \ \forall\ x \in A$. For every $n \in \mathbb{Z}_0$, let $\mathcal{P}_n := \mathcal{P}_n[\mathbb{R}]$ and $\mathcal{P}_n[\mathbb{C}]$ denote the sets of all algebraic polynomials of degree at most n with real and complex coefficients, respectively, and let also $\mathcal{P} := \bigcup_{n\in\mathbb{Z}_0} \mathcal{P}_n$, $\mathcal{P}[\mathbb{C}] := \bigcup_{n\in\mathbb{Z}_0} \mathcal{P}_n[\mathbb{C}]$. If $\varphi : \mathbb{R} \to \mathbb{R}$ then the function $M_\varphi(x) := \lim\limits_{\delta\downarrow 0} \sup\limits_{y\in(x-\delta,x+\delta)} \varphi(y)$ is called an upper Bair function of φ , and $\delta \downarrow 0$ means $\delta \to 0$ and $\delta > 0$.

To shorten expressions the following notations will be used:

$$I := [-1, 1];\ \ I_0 := (-1, 1);\ \ J := (-\infty, -1) \cup (1, +\infty),\ \ I_R := R \cdot I,\ \ J_R := R \cdot J,\ \ R > 0\ .$$

For nonnegative function $F : \mathbb{R} \to \mathbb{R}^+ := [0, +\infty)$ let $S_F := \{\, x \in \mathbb{R} \mid F(x) > 0\}$.

Let $\mathfrak{B}(\mathbb{R})$ denote the family of Borel subsets of $\mathbb{R}$, $\mathcal{M}(\mathbb{R})$ - linear space of finite Borel measures on $\mathbb{R}$ and $L_p(\mu) := L_p(\mathbb{R}, d\mu)$, $\|f\|^p_{L_p(\mu)} := \int\limits_{\mathbb{R}} |f(x)|^p\ d\mu(x)$, $1 \leq p < \infty$. It should be reminded that every measure $\mu \in \mathcal{M}(\mathbb{R})$ is regural [6, VI, Def.8.2, Ex.8.16] and therefore for any positive $\mu \in \mathcal{M}(\mathbb{R})$ and arbitrary $A \in \mathfrak{B}(\mathbb{R})$ there exists [6, VI, (8.14)] such sequence of compactly supported continuous functions $\psi_n[A, \mu] : \mathbb{R} \to [0, 1]$, $n \geq 1$, that

$$\lim_{n\to\infty} \|\ \chi_A - \psi_n[A, \mu]\ \|_{L_p(\mu)} \ = \ 0 \ \ \forall\ 1 \leq p < \infty\ . \tag{1.1.1}$$

For every $\mu \in \mathcal{M}(\mathbb{R})$ Hahn expansion of the space $(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$ with respect to the measure μ will be denoted by $\mathbb{R} = \mathbb{R}^+_\mu \sqcup \mathbb{R}^-_\mu$, where $A \sqcup B$ denotes union of disjoint sets A and B [6, I, Th.16.2]. For the expansion of the measure $\mu \in \mathcal{M}(\mathbb{R})$ in the sense of Jordan we will use the following notations: $\mu = \mu_+ - \mu_-$, $\mu_+(A) := \mu(A \cap \mathbb{R}^+_\mu)$, $\mu_-(A) := \mu(A \cap \mathbb{R}^-_\mu)$, $\forall A \in \mathfrak{B}(\mathbb{R})$, and $\|\mu\| := |\mu|(\mathbb{R})$, where $|\mu| := \mu_+ + \mu_-$ [6, I.16].

1.2. Background. For arbitrary $w : \mathbb{R} \to [0,1]$ consider the seminormed space

$$C_w^0 := \left(\left\{ \ f \in C(\mathbb{R}) \mid \lim_{|x|\to\infty} w(x)f(x) \ = 0 \ \right\} \ , \quad \|\cdot\|_w \ \right), \tag{1.2.1}$$

where $\|f\|_w := \|w \cdot f\|_{C(S_w)} \quad \forall \ f \in C_w^0$. An obvious inequality $\|f\|_w \leq \|f\|_{C(\mathbb{R})}$ $\forall \ f \in C^0(\mathbb{R})$ implies inclusions $C^0(\mathbb{R}) \subseteq C_w^0 \subset C(\mathbb{R})$ and validity of the continuous embedding $C^0(\mathbb{R}) \hookrightarrow C_w^0$ [13, 0.2.9]. Since [13, IV, Ex.4.45] for every $L \in C^0(\mathbb{R})^*$ there exists such $\mu \in \mathcal{M}(\mathbb{R})$ that $L(f) = \int_{\mathbb{R}} f(x) \ d\mu(x) \quad \forall \ f \in C^0(\mathbb{R})$, indicated embedding means that for any element L of the Banach space $(C_w^0)^*$ with norm $\|L\|_w :=$ $\sup\{|L(f)| \mid f \in C_w^0, \|f\|_w \leq 1\}$ there exists such $\mu_L \in \mathcal{M}(\mathbb{R})$ that

$$L(f) = \int_{\mathbb{R}} f(x) \ d\mu_L(x) \quad \forall \ f \in C^0(\mathbb{R}) \ . \tag{1.2.2}$$

In this Chapter, we describe the Banach space B_w^0 being a completion of the normed space N_w^0 associated [13, I.10.2] with seminormed space C_w^0 , give a supplement to S. Mergelyan's Theorem [32, Th.7], formulate the weighted analog of the Weierstrass polynomial approximation theorem and establish a general form of any functional in $(C_w^0)^*$, i.e. in view of (1.2.2) find a complete description of the subspace $\{\mu_L\}_{L\in(C_w^0)^*} \subseteq$ $\mathcal{M}(\mathbb{R})$.

Remark 1.1. (Mergelyan's regularity.) Studying the polynomial approximation problem in C_w^0 S. Mergelyan suggested [32] to change the weight function w by its upper Bair function M_w . Let us clarify what does that suggestion mean in terms of the seminormed spaces. It is known [33] that an upper Bair function M_w be an upper semicontinuous function [17] and the following relations hold:

$$0 \leq w(x) \leq \ M_w(x) \leq 1 \quad \forall \ x \in \mathbb{R} \ ; \quad S_w \subseteq S_{M_w} \subseteq \overline{S}_w = \overline{S}_{M_w} \ , \tag{1.2.3}$$

Besides that for any open set $G \subseteq \mathbb{R}$:

$$\|f \cdot \chi_G\|_w \ = \ \|f \cdot \chi_G\|_{M_w} \quad \forall \ f \in C(\overline{S}_w) \ . \tag{1.2.4}$$

Therefore the seminormed spaces C_w^0 and $C_{M_w}^0$ coincide identically, i.e. $C_w^0 \equiv C_{M_w}^0$. In spite of the available possibility to consider everywhere below only upper semicontinuous functions w , i.e. $w = M_w$, we will not do so and will examine a general case $w \neq M_w$ using notation:

$$h := M_w \ .$$

1.3. Associated normed space N_w^0 . One can easily conclude from known criterion [13, 1.10.1] of the separability of locally convex spaces and from the continuity of functions in C_w^0 that seminormed space C_w^0 is a normed one if and only if $\overline{S}_w = \mathbb{R}$.

Denote by $N_w^0 := C_w^0 \setminus N_{C_w^0}$ (see 1.1) the normed space associated with C_w^0 . Introduce the normed space

$$C_w^0(\overline{S}_w) := \left(\left\{ \ f \in C(\overline{S}_w) \mid \lim_{x\in\overline{S}_w, \ |x|\to\infty} w(x)f(x) \ = \ 0 \ \right\}, \ \|\cdot\|_w \ \right) \tag{1.3.1}$$

and corresponding two normed ones of the restrictions: $C_w^0(\overline{S}_w)\restriction_{S_h}:= (\ \{\ f\restriction_{S_h}\ \ |\ \ f\in$ $\in\ C_w^0(\overline{S}_w)\}\ ,\|\cdot\|_w)\ \ ,\ C_w^0(\overline{S}_w)\restriction_{S_w}:=\big(\{f\restriction_{S_w}|f\in C_w^0(\overline{S}_w)\}\ ,\|\cdot\|_w\big)$. Due to (1.2.1), (1.2.4) $C_w^0(\overline{S}_w)\equiv C_h^0(\overline{S}_h)$ and hence, $C_w^0(\overline{S}_w)\restriction_{S_h}\equiv C_h^0(\overline{S}_h)\restriction_{S_h}$. It's evident, that transformations $f\to f\restriction_{S_h}\ ,\ f\to f\restriction_{S_w}\ \ \forall\ f\in C_w^0(\overline{S}_w)$ determine isometric relations $C_w^0(\overline{S}_w)\simeq C_h^0(\overline{S}_h)\restriction_{S_h}$ and $C_w^0(\overline{S}_w)\simeq C_w^0(\overline{S}_w)\restriction_{S_w}$, respectively. Besides that defined by formula $V(\pi(f))=f\restriction_{\overline{S}_w}$ $\forall\ f\in C_w^0$ transformation

$$V\ :\ N_w^0\ \to\ C_w^0(\overline{S}_w)\ , \tag{1.3.2}$$

determines linear isometry (see 1.1 and [3, IV.1.3]) of the spaces N_w^0 and $C_w^0(\overline{S}_w)$. Really, equality $\|V(\pi(f))\|_w\ =\ \|\pi(f)\|_{N_w^0}\ \equiv\ \|f\|_w$ is obvious and relation $V(N_w^0)\ =\ C_w^0(\overline{S}_w)$ follows from known continuity [33, IV.4, Lemma 2] of the linear extension f_c to the whole real line of some continuous on the closed set $F\subset\mathbb{R}$ function f . In addition if [4, IV.5, Th.21] interval of the kind $(-\infty,b)$ or $(a,+\infty)$ is a part of $\mathbb{R}\setminus F$ then we will regard:

$$\begin{cases} f_c(b-\theta):=\theta f(b) & \forall\ \theta\in(0,1);\\ f_c(b-\lambda):=0 & \forall\ \lambda\ \geq 1\ ;\end{cases}\quad \begin{cases} f_c(a+\theta):=\theta f(a) & \forall\ \theta\in(0,1);\\ f_c(a+\lambda):=0 & \forall\ \lambda\ \geq 1\ ;\end{cases} \tag{1.3.3}$$

respectively. Therefore

$$N_w^0\cong C_w^0(\overline{S}_w)\equiv C_h^0(\overline{S}_h)\cong C_h^0(\overline{S}_h)\restriction_{S_h}\cong C_w^0(\overline{S}_w)\restriction_{S_w}\ , \tag{1.3.4}$$

i.e. associated with C_w^0 normed space N_w^0 can be identified with arbitrary indicated in (1.3.4) isometric normed spaces.

1.4. Banach space B_w^0 .

Definition 1.4.1. *Let* $w:\mathbb{R}\to[0,1]$, $h:=M_w$ *is an upper Bair function of* w *and* $S_h:=\{x\in\mathbb{R}\mid h(x)>0\ \}$. *The space* B_w^0 *is called a Banach space associated with the seminormed space* C_w^0 *if* B_w^0 *is equipped with norm* $\|f\|_h:=\sup\limits_{x\in S_h}h(x)|f(x)|$ *and consists of all functions* $f:S_h\to\mathbb{R}$, *which satisfy the following three conditions:*

(1.4.1) *function* f *is continuous on the set* $E_{1/\delta}(h):=h^{-1}([\delta,1])=$ $=\{x\in\mathbb{R}\mid h(x)\geq\delta\}$ *for any* $\delta\in(0,1]$;

(1.4.2) $\lim\limits_{h(x)\to 0}h(x)f(x)=0$, i.e.

$$\forall\varepsilon>0\ \exists\ \delta>0\ :\ \{x\in\mathbb{R}\mid 0<h(x)<\delta\ \}\subseteq\{x\in S_h\mid h(x)|f(x)|<\varepsilon\ \}\ ;$$

(1.4.3) $h(x)f(x)\to 0$, $x\in S_h$, $|x|\to\infty$.

If $\lim\limits_{|x|\to\infty}h(x)\ =\ 0$, then property (1.4.2) implies (1.4.3) and in that case property (1.4.3) can be excluded from Definition 1.4.1.

Verify now correctness of the Definition 1.4.1.

Since h is an upper semicontinuous function then [17] all sets $E_{1/\delta}(h)$ for $\delta\in(0,1]$, are closed and so for any $f\in B_w^0$ and $\varepsilon=1$ one can find such $R>0$ in (1.4.3) and $\delta>0$ in (1.4.2) that $\|h\cdot f\|_{C(J_R)}\leq 1$, $\|h\cdot f\|_{C(h^{-1}(0,\delta))}\leq 1$, getting on the supplement

compact $S_h \setminus [J_R \cup h^{-1}(0,\delta)] = I_R \cap E_{1/\delta}(h)$ the uniformly boundedness f by property (1.4.1). That's why $\|f\|_h < \infty \ \ \forall\ f \in B_w^0$.

If now $\{f_n\}_{n\geq 1} \subset B_w^0$ is a fundamental sequence in B_w^0 then by known scheme [4, V.5] one can easily obtain an existence of such $F: S_h \to \mathbb{R}$ that $\lim_{n\to\infty} \|f_n - F\|_h = 0$. Function F obviously satisfies conditions (1.4.2) and (1.4.3). Since for every $m \geq 1$ and $x \in E_m(h)\ :\ h(x) \geq \frac{1}{m}$, then $\|f_n - F\|_h \geq \frac{1}{m}\|f_n - F\|_{C(E_m(h))}$, i.e. the sequence $f_n \restriction_{E_m(h)} \in C(E_m(h))$, $n \geq 1$, uniformly on the set $E_m(h)$ converges to $F \restriction_{E_m(h)}$ and therefore [4, IV.2] $F \restriction_{E_m(h)} \in C(E_m(h))$. That's why $F \in B_w^0$ and introduced in Definition 1.4.1 normed space be in fact Banach one.

It should be noted at last that as well as C_w^0 (see Remark 1.1) Banach space B_w^0 posseses the property $B_w^0 \equiv B_{M_w}^0$. Let now formulate the basic result of that section.

THEOREM 1.1. *Let $w: \mathbb{R} \to [0,1]$, $h := M_w$ is an upper Bair function of w and $S_h := \{x \in \mathbb{R} \mid h(x) > 0\ \}$. Linear operator*

$$T: C_w^0 \ \to\ B_w^0\ ,$$

defined by formula

$$Tf = f \restriction_{S_h} \quad \forall\ f \in C_w^0 \tag{1.4.4}$$

isometrically and tightly embeds seminormed space C_w^0 into the Banach space B_w^0 , i.e.

(1.4.5a) $\|Tf\|_h = \|f\|_w \ \ \forall\ f \in C_w^0$;

(1.4.5b) $T(C_w^0)$ *is a dense subspace of the Banach space* B_w^0 .

In addition $T(C_w^0)$ coincides with the subspace of those functions $f \in B_w^0$, which can be extended to the continuous on $\overline{S}_h$ function.

Validity of the following implication (see 1.7.2):

$$\begin{aligned} &\exists\, \{x_n\}_{n\geq 0} \subseteq S_h\ :\ \ \lim_{n\to\infty} x_n = x_0 \in \mathbb{R},\ \lim_{x_n\to x_0} h(x_n) = 0\ \Rightarrow \\ &\Rightarrow\ \exists F \in B_w^0\ :\qquad \|F\|_{C(S_h\cap(x_0-\delta,x_0+\delta))}\ =\ +\infty\ \ \forall\ \delta\ >\ 0 \end{aligned} \tag{1.4.6}$$

allows us to characterize the following partial cases of Theorem 1.1.

COROLLARY 1.1. *Let $w: \mathbb{R} \to [0,1]$, $h := M_w$. Then*

(1.4.7) C_w^0 *is a normed space if and only if* $\overline{S}_h = \mathbb{R}$;

(1.4.8) *The following statements are equivalent:*

(1.4.8*a*) C_w^0 *is a Banach space* ; (1.4.8*b*) $B_w^0 = C_w^0$;
(1.4.8*c*) $S_h = \mathbb{R}$ *and* $B_w^0 \subseteq C(\mathbb{R})$; (1.4.8*d*) $\inf_{x\in[-R,R]} h(x)\ >\ 0 \quad \forall\ R > 0$;

(1.4.9) *The following statements are equivalent:*

(1.4.9*a*) N_w^0 *is a Banach space* ; (1.4.9*b*) $B_w^0 = C_w^0(\overline{S}_w)$;
(1.4.9*c*) $S_h = \overline{S}_h$ *and* $B_w^0 \subseteq C(\overline{S}_h)$; (1.4.9*d*) $\inf_{x\in S_h\cap[-R,R]} h(x)\ >\ 0 \quad \forall\ R > 0$;

where $\inf \emptyset := +\infty$.

The following application of the Theorem 1.1 gives some explanation why everywhere above we have not assumed the upper semicontinuity of w .

Corollary 1.2. *Let* $w : \mathbb{R} \to [0,1]$ *and* $\mathcal{M}$ *is some dense subset of the seminormed space* C_w^0 *. Function* $f : S_w \to \mathbb{R}$ *can be approximated by elements of* $\mathcal{M}$ *, i.e.*

$$\forall \varepsilon > 0 \ \exists \ m_\varepsilon \in \mathcal{M} \ : \quad w(x)\,|f(x) - m_\varepsilon(x)| < \varepsilon \quad \forall \ x \in S_w \tag{1.4.10}$$

if and only if $\exists g \in B_w^0 : \ g\restriction_{S_w} = f$.

1.5. Supplement to S.N. Mergelyan's Theorem . In [32, Th.7] S. Mergelyan proved that for the weight function $w : \mathbb{R} \to [0,1]$ satisfying condition

$$\|x^n w\|_{C(\mathbb{R})} < +\infty \quad \forall \ n \in \mathbb{Z}_0 \ , \tag{1.5.1}$$

either algebraic polynomials $\mathcal{P}$ are dense in C_w^0 or they can approximate only those functions $f : S_w \to \mathbb{R}$ which can be extended from their domain of definition S_w into the whole complex plane as an entire function of minimal exponential type. I. Hachatryan [14] gived the description of the indicated in the Mergelyan's theorem class of entire functions. Corollary 1.2 implies the following supplement to the Mergelyan's theorem when algebraic polynomials $\mathcal{P}$ are dense in C_w^0 .

Theorem 1.2. *Let* $w : \mathbb{R} \to [0,1]$ *satisfies conditions (1.5.1),* M_w *be an upper Bair function of* w *and algebraic polynomials* $\mathcal{P}$ *are dense in* C_w^0 *. Then the function* $f : S_w \to \mathbb{R}$ *can be approximated by polynomials, i.e.*

$$\exists \{P_n\}_{n\geq 1} \subset \mathcal{P} \ : \lim_{n\to\infty} \sup_{x\in S_w} w(x)\,|P_n(x) - f(x)| \ = \ 0 \ ,$$

if and only if that function can be extended into the set S_{M_w} *as a function* $f : S_{M_w} \to \mathbb{R}$ *, satisfying the following conditions:*

(1.5.2) *for every* $m \geq 1$ f *is a continuous function on the closed set* $\left\{x \in \mathbb{R} \mid M_w(x) \geq \frac{1}{m}\right\}$;

(1.5.3) $\lim_{M_w(x)\to 0} M_w(x) \cdot f(x) = 0$, i.e. $\forall \ \varepsilon > 0 \ \exists \ \delta > 0$:

$$\{x \in \mathbb{R} \mid 0 < M_w(x) < \delta\} \subseteq \{x \in S_{M_w} \mid M_w(x) \cdot |f(x)| < \varepsilon\} \ .$$

If S_w is a bounded set then conditions (1.5.1) are obviously true and by Weierstrass approximation theorem algebraic polynomials $\mathcal{P}$ are dense in C_w^0 . That's why for arbitrary weight $w : \mathbb{R} \to [0,1]$ with bounded set S_w conditions (1.5.2) and (1.5.3) give the

weighted analog of the Weierstrass polynomial approximation theorem. It is interesting to note also that for the weight $w(x) = \sqrt{1-x^2} \cdot \chi_{[-1,1]}(x)$ conditions (1.5.2) and (1.5.3) are equivalent to $f \in C((-1,1))$ and $\lim\limits_{|x|\to 1} \sqrt{1-x^2} f(x) = 0$. This fact is known and can be found in [28] where according to these two conditions the subspaces of known spaces B^r were introduced.

1.6. General form of the functionals in $(C_w^0)^*$. To prove the main theorem of that section the following version of known M. Krein's lemma will be necessary [37].

LEMMA 1.1. *Let (X,p) be a seminormed space and $K \subset X$ is a normal cone, i.e. convex set K satisfies: $\lambda \cdot K \subseteq K \;\; \forall \lambda \geq 0$ and $p(x) \leq p(x+y) \;\; \forall\, x,y \in K$. If X^* is a Banach space conjugate to X then for $K^* := \{x^* \in X^* \mid x^*(x) \geq 0 \;\; \forall x \in K \}$ the following equality holds:*

$$K^* - K^* = X^*$$

It should be noted that in [5] a necessary and sufficient condition for the validity of more general equality $(K_1 \cap K_2)^* = K_1^* + K_2^*$ has been established and a notion of normal pair of cones (K_1, K_2) of transfinite order α has been introduced. Now we can formulate the main theorem of this section.

THEOREM 1.3. *Let $w : \mathbb{R} \to [0,1]$, M_w is an upper Bair function of w and $S_{M_w} := \{x \in \mathbb{R} \mid M_w(x) > 0 \}$. If L is a linear continuous functional on the seminormed space C_w^0 then there exists such measure $\mu \in \mathcal{M}(\mathbb{R})$, that $|\mu|(\mathbb{R} \setminus S_{M_w}) = 0$ and*

$$L(f) = \int\limits_{\mathbb{R}} M_w(x) f(x)\; d\mu(x) \;\; \forall\; f \in C_w^0 . \tag{1.6.1}$$

For arbitrary $\mu \in \mathcal{M}(\mathbb{R})$ defined by formula (1.6.1) functional L is linear and continuous on the seminormed space C_w^0 with $\|L\| = |\mu|(S_{M_w})$ (see 1.1).

1.7. Proofs.

1.7.1. PROOF OF THEOREM 1.1. Equality (1.4.5a) follows from (1.2.4). Prove now that $T(C_w^0) \subseteq B_w^0$, where $T(C_w^0) \equiv \{f \restriction_{S_h} \mid f \in C_w^0 \} =: C_w^0 \restriction_{S_h}$. If $g = f \restriction_{S_h}$, $f \in C_w^0$, then conditions (1.4.1) and (1.4.3) for function g are realized. Let us prove that g satisfies property (1.4.2). For given $\varepsilon > 0$ one can find by (1.4.3) such $R(\varepsilon) > 0$ that: $\|hg\|_{C(S_h \cap J_{R(\varepsilon)})} < \varepsilon$. Denote $C(\varepsilon) := \|g\|_{C(S_h \cap I_{R(\varepsilon)})} \leq \|f\|_{C(\overline{S}_h \cap I_{R(\varepsilon)})} < \infty$. Then the number $\delta(\varepsilon) := \varepsilon / C(\varepsilon) > 0$ in view of $\|hg\|_{C(h^{-1}(0,\delta(\varepsilon)) \cap I_{R(\varepsilon)})} \leq \delta(\varepsilon) C(\varepsilon) = \varepsilon$ will be required for the validity of (1.4.2), i.e. $g \in B_w^0$. In addition proved in 1.3 equality $C_w^0 \restriction_{\overline{S}_h} = C_w^0(\overline{S}_h)$ yields

$$T(C_w^0) \equiv C_w^0 \restriction_{S_h} = C_w^0(\overline{S}_h) \restriction_{S_h} \subseteq B_w^0 , \tag{1.7.1.1}$$

and to finish the proof it is remained to show that $C_w^0 \restriction_{S_h}$ is dense in B_w^0 .

Consider an arbitrary $f \in B_w^0$, $\varepsilon > 0$ and prove that there exists such $f_\varepsilon \in C_w^0$ that $\|f - f_\varepsilon\|_h \le 2\varepsilon$. Property (1.4.2) admits to find such positive integer $m \ge 1$ that

$$\begin{cases} \left\{ x \in \mathbb{R} \mid 0 < h(x) < \frac{1}{m} \right\} \subseteq \{x \in S_h \mid h(x)|f(x)| < \varepsilon\} \ ; \\ m > \frac{\|f\|_h}{\varepsilon} \ , \quad E_m \ne \emptyset \ , \end{cases} \tag{1.7.1.2}$$

where (see (1.4.1)) $E_p := E_p(h) \ \ \forall\, p \ge 1$.

Since f is a continuous function on the set E_{m^2} , then by means of indicated in 1.2 method we extend f into the whole real line obtained $f_\varepsilon \in C_w^0$. Let us prove that

$$\|f - f_\varepsilon\|_h \le 2\varepsilon \ . \tag{1.7.1.3}$$

Since for $x \in S_h \setminus E_{m^2}$: $0 < h(x) < \frac{1}{m^2} \le \frac{1}{m}$, then by (1.7.1.2): $\|f - f_\varepsilon\|_h = \|h(f - f_\varepsilon)\|_{C(S_h \setminus E_{m^2})} \le \varepsilon + \|hf_\varepsilon\|_{C(S_h \setminus E_{m^2})}$ and therefore for the validity of (1.7.1.3) it is sufficient to prove that

$$\|hf_\varepsilon\|_{C(\mathbb{R} \setminus E_{m^2})} \ \le \ \varepsilon \ . \tag{1.7.1.4}$$

Inequality (1.7.1.4) is trivial if $E_{m^2} = \mathbb{R}$. But if $\mathbb{R} \setminus E_{m^2} \ne \emptyset$ then $\mathbb{R} \setminus E_m \ne \emptyset$ and by (1.7.1.2) $\mathbb{R} \setminus E_m \ne \mathbb{R}$. Consider an arbitrary $x \in \mathbb{R} \setminus E_{m^2} \subseteq \mathbb{R} \setminus E_m$. Such x belongs to one of the forming [4, IV.5, Th.21] open set $\mathbb{R} \setminus E_m$ interval (a, b) and moreover $x \in (a_k, b_k) \subseteq (a, b) \setminus E_{m^2}$, where $(a, b) \setminus E_{m^2} = \bigsqcup_{k=1}^{Q} (a_k, b_k)$, $a_k, b_k \in E_{m^2} \cup \{\pm\infty\}$, $1 \le Q \le \infty$.

Assume that (a_k, b_k) is a bounded interval. Then $\exists \theta \in (0, 1) : x = \theta a_k + (1 - \theta) b_k$, and

$$h(x)|f_\varepsilon(x)| \ \le \ \theta h(x)|f(a_k)| + (1 - \theta) h(x)|f(b_k)| \ . \tag{1.7.1.5}$$

If $a_k > a$, then $a_k \in \mathbb{R} \setminus E_m$, $h(x) < \frac{1}{m^2} \le h(a_k) < \frac{1}{m}$ and by (1.7.1.2) $h(x)|f(a_k)| \le h(a_k)|f(a_k)| \le \varepsilon$. If $a_k = a$ then $a_k \in E_m$ and $h(x) < \frac{1}{m^2} \le \frac{h(a_k)}{m}$, whence in view of (1.7.1.2): $h(x)|f(a_k)| \le \frac{1}{m} h(a_k)|f(a_k)| \le \frac{\|f\|_h}{m} \le \varepsilon$. Performing the same estimate for $h(x)|f(b_k)|$ we obtain from (1.7.1.5) $h(x)|f_\varepsilon(x)| \le \varepsilon$.

Assume now that $(a_k, b_k) = (a_k, +\infty)$. Then by (1.3.3) for $x = a_k + \theta$, $\theta \in (0, 1)$: $h(x)|f_\varepsilon(x)| = \theta h(x)|f(a_k)|$, and for $x \ge a + 1$: $f_\varepsilon(x) = 0$. Since $E_m \ne \emptyset$, then $(a, b) \ne \mathbb{R}$ and hence, in that case $(a, b) = (a, +\infty)$. Estimated $h(x)|f(a_k)|$ as well as it has been done above for the case of bounded interval (a_k, b_k) we will get $h(x)|f_\varepsilon(x)| \le \varepsilon$ again.

The case $(a_k, b_k) = (-\infty, b_k)$ can be considered in just the same way. That's why inequality (1.7.1.4) together with Theorem 1.1 is proved.

1.7.2. Proof of implication (1.4.6). Without loss of generality one can consider that the sequences $\frac{1}{\lambda_n} := h(x_n) > 0$, and $|x_n - x_0|$, $n \in \mathbb{Z}_0$, are decreasing. Let as in 1.7.1 $E_\lambda := E_\lambda(h)$ for $\lambda \in [1, +\infty)$. Since $x_{n+1} \in \mathbb{R} \setminus E_{\lambda_n}$ $\forall n \in \mathbb{Z}_0$, it is possible to find such sequence $\{\delta_n\}_{n \ge 1}$ of positive real numbers that $x_{n+1} + \delta_{n+1} I_0 \subset \mathbb{R} \setminus E_{\lambda_n}$ $\forall n \in \mathbb{Z}_0$ and the sets $\{x_n + \delta_n I\}_{n \ge 1}$ are disjoint. Set

$$F(x) := \sum_{k \ge 1} \sqrt{\lambda_{k-1}}\, \alpha_k(x) \ , \quad \alpha_k(x) = \left(1 - \left|\frac{x - x_k}{\delta_k}\right|\right) \chi_I \left(\frac{x - x_k}{\delta_k}\right) \ , k \ge 1 \ . \tag{1.7.2.1}$$

Equalities $F(x_k) = \sqrt{\lambda_{k-1}}\ \ \forall k \geq 1$ imply $\|F\|_{C(S_h \cap (x_0-\delta, x_0+\delta))} = +\infty\ \ \forall \delta > 0$. It remains to prove that $F \in B^0_w$. Since $x_{n+1+p} + \delta_{n+1+p} I_0 \subseteq \mathbb{R} \setminus E_{\lambda_{n+p}} \subseteq \mathbb{R} \setminus E_{\lambda_n}\ \ \forall n, p \in \mathbb{Z}_0$ then for every $n \geq 1$:

$$F(x) \restriction_{E_{\lambda_n}} = \sum_{k=1}^{n} \sqrt{\lambda_{k-1}} \alpha_k(x) \restriction_{E_{\lambda_n}} \in\ C(E_{\lambda_n}) ,$$

and consequently, property (1.4.1) for F is fulfilled. Validity of (1.4.3) is obviuos. Let us show that F satisfies (1.4.2). Really for any $n \geq 1$ inequalities $0 < h(x) < \frac{1}{\lambda_n}$ yield

$$h(x)F(x) \leq \sum_{k \geq 1} \alpha_k(x) \min \left\{ \frac{1}{\sqrt{\lambda_{k-1}}} ,\ \frac{\sqrt{\lambda_{k-1}}}{\lambda_n} \right\} \leq \frac{1}{\sqrt{\lambda_n}} ,$$

what means the validity of (1.4.2). That is why $F \in B^0_w$ and implication (1.4.6) is proved.

1.7.3. Proof of Corollary 1.1. Correctness of (1.4.7) was proved in 1.2. Implications (1.4.8b)⇒(1.4.8a), (1.4.8c)⇒(1.4.8b), (1.4.9c)⇒(1.4.9b) are evident and (1.4.9b)⇒(1.4.9a) follows from (1.3.4).

(1.4.8a)⇒(1.4.8c). Since C^0_w is a normed space then by (1.4.7) $\mathbb{R} = \overline{S}_w = \overline{S}_h$ (see (1.2.3)). Besides that by Theorem 1.1 $T(C^0_w) = C^0_w \restriction_{S_h} = B^0_w$. Then assumption $\overline{S}_h \setminus S_h \neq \emptyset$ together with upper semicontinuity h and (1.4.6) leads to a contradiction. That' why $\mathbb{R} = \overline{S}_h = S_h$ and $B^0_w = C^0_w \subset C(\mathbb{R})$.

(1.4.8c)⇒(1.4.8d). If $\exists R > 0 : \inf_{x \in [-R,R]} h(x) = 0$, then (1.4.6) yields $S_h \neq \mathbb{R}$ or $B^0_w \setminus C(\mathbb{R}) \neq \emptyset$ in the case $S_h = \mathbb{R}$.

(1.4.8d)⇒(1.4.8c). If for any $R > 0 : 1/\lambda(R) := \inf_{x \in I_R} h(x) > 0$, then $S_h = \mathbb{R}$ and $I_R \subseteq E_{\lambda(R)}(h)\ \ \forall R > 0$. Therefore for arbitrary $f \in B^0_w : f \in C(I_R)\ \ \forall R > 0$, and hence, $B^0_w \subset C(\mathbb{R})$.

(1.4.9a)⇒(1.4.9c). Due to (1.3.4) and (1.7.1.1): $B^0_w = C^0_w(\overline{S}_h) \restriction_{S_h}$. An upper semicontinuity of h and (1.4.6) give $\overline{S}_h = S_h$ and hence, $B^0_w = C^0_w(\overline{S}_h) \subseteq C(\overline{S}_h)$.

(1.4.9c)⇒(1.4.9d). If $\exists R > 0 : S_h \cap I_R \neq \emptyset$ and $\inf_{x \in S_h \cap I_R} h(x) = 0$, then $S_h \neq \overline{S}_h$ or $B^0_w \setminus C(\overline{S}_h) \neq \emptyset$ in the case $S_h = \overline{S}_h$ by the property (1.4.6).

(1.4.9d)⇒(1.4.9c). Let $1/\lambda(R) := \inf_{x \in I_R \cap S_h} h(x) > 0\ \ \forall R \geq R_0$, where $S_h \cap I_{R_0} \neq \emptyset$. Then for such values of R : $I_R \cap S_h \subseteq E_{\lambda(R)}(h)$ and by the closure of $E_{\lambda(R)}(h)$: $I_R \cap \overline{S}_h \subseteq E_{\lambda(R_1)}(h) \subseteq S_h\ \ \forall R_1 > R \geq R_0$. Therefore $S_h = \overline{S}_h$ and for any $f \in B^0_w$: $f \in C(I_R \cap \overline{S}_h)\ \ \forall R > R_0$. That is why $B^0_w \subseteq C(\overline{S}_h)$ and Corollary 1.1 is proved.

1.7.4. Proof of Corollary 1.2.

Necessity. If some $f : S_w \to \mathbb{R}$ satisfies (1.4.10) then for $\varepsilon = \frac{1}{p}$, $p \geq 1$, we obtain from there fundamental sequence $\{m_{1/p}\}_{p \geq 1} \subseteq C^0_w \equiv C^0_h$, $h := M_w$, being mapped by transformation (1.4.4) onto the fundamental sequence in Banach space B^0_w whose limit $g \in B^0_w$ due to (1.2.3) satisfies: $g \restriction_{S_w} = f$.

Sufficiency. Assume that $\exists g \in B^0_w$: $g \restriction_{S_w} = f$. Then by Theorem 1.1 the set $\mathcal{M} \restriction_{S_h} \equiv T(\mathcal{M})$ will be dense in Banach space B^0_w . That is why those elements of $\mathcal{M} \restriction_{S_h}$ which approximate g in B^0_w in view of (1.2.3) will approximate f on $S_w \subseteq S_h$ in the sense of (1.4.10). Corollary 1.2 is proved.

1.7.5. Proof of Lemma 1.1. It is easy to verify that the closure $\overline{K}$ in X will be a normal cone and $(\overline{K})^* = K^*$. That is why we may consider that K is a closed cone.

Following well-known scheme of [37, I, Ex.2a] let us examine any subspace $Y \subset X$ which is an algebraic complementary subspace to $N := \{x \in X \mid p(x) = 0\}$ and for arbitrary $x \in X$ in its representation $x = n + y$, $n \in N$, $y \in Y$, denote $P_Y x := y$. It follows from the closure of K and an obvious equality

$$p(x+n) = p(x) \ \ \forall\, n \in N \ \ \forall x \in X \ , \tag{1.7.5.1}$$

that

$$P_Y(K) = K \cap Y \ , \quad K = N + K \cap Y \ , \tag{1.7.5.2}$$

and cone $K \cap Y$ is a normal one in the normed space (Y, p) . Thus, by M. Krein's lemma (see[37])

$$Y^* = (Y \cap K)^* - (Y \cap K)^* \ . \tag{1.7.5.3}$$

Now for any $L \in X^*$ equality $\|L\|_{X^*} := \sup\{|L(x)| \mid x \in X,\, p(x) \le 1\}$ implies $L(x) = 0$ $\forall\, x \in N$ and so defined by formula $l(y) := L(y)$ $\forall\, y \in Y$ functional l will be an element of Y^* . According to (1.7.5.3) $l = l_1 - l_2$, $l_i \in Y^*$, $l_i(Y \cap K) \ge 0$, $i \in \{1,2\}$. Extending each functional l_i onto the whole space X by formula $L_i(x) := l_i(P_Y x)$ with the help of (1.7.5.1) we get $L_i \in X^*$ and $0 \le L_i(Y \cap K) = L_i(N + Y \cap K) \overset{(1.7.5.2)}{=} L_i(K)$, $i \in \{1,2\}$. That is why $L = L_1 - L_2,\ L_1, L_2 \in K^*$, as was to be proved.

1.7.6. Proof of Theorem 1.3. Since $C^0_w \equiv C^0_{M_w}$ then it is sufficient to prove the statement of theorem only in the case when function w is upper semicontinuous on $\mathbb{R}$.

Let $L \in (C^0_w)^*$ and K be a cone of all nonnegative on the real line functions from C^0_w , which is a normal one in the seminormed space C^0_w . Using Lemma 1.1 we can find such $L_+, L_- \in (C^0_w)^*$, that

$$L = L_+ - L_- \ , \ L_+(K) \ge 0,\ L_-(K) \ge 0 \ . \tag{1.7.6.1}$$

Formula (1.2.2) allows us to find measures $\mu_L, \mu_L^+ \mu_L^- \in \mathcal{M}(\mathbb{R})$, relevant to the functionals L, L_+, L_- . Exploiting regularity of the measures in $\mathcal{M}(\mathbb{R})$ and density of all compactly supported continuous functions in the spaces $L_1(\nu)$, $\nu \in \left\{\mu_L^+, \mu_L^-, \mu_L^+ + \mu_L^-\right\}$ (see 1.1), it is easy to verify that $\mu_L = \mu_L^+ - \mu_L^-$ and measures μ_L^+, μ_L^- are positive.

Consider now an arbitrary measure $\nu \in \left\{\mu_L^+, \mu_L^-\right\}$ and corresponding functional $L_\nu \in \{L^+, L^-\}$, that for any $f \in C^0(\mathbb{R})$:

$$L_\nu(f) = \int\limits_{\mathbb{R}} f(x) d\nu(x); \qquad \|L_\nu\| := \sup\{|L_\nu(f)| \ \big| f \in C^0_w,\ ||f||_w \le 1\} < \infty. \tag{1.7.6.2}$$

Since for arbitrary $\varepsilon > 0$ function $1/(\varepsilon + w(x))$ is lower semicontinuous then using the known fact from [17, I, Th.1.4] we get a nondecreasing sequence of positive and continuous on the whole real axis functions $\varphi_n^\varepsilon(x),\ n \ge 1 :\ \lim_{n\to\infty} \varphi_n^\varepsilon(x) = 1/(\varepsilon + w(x))$ $\forall x \in \mathbb{R}$. Setting

$$w_n^\varepsilon(x) := e^{-\frac{x^2}{n}} \varphi_n^\varepsilon(x) \ , \ n \ge 1,\ x \in \mathbb{R} \ , \tag{1.7.6.3}$$

and taking into account $\|\varphi_n^\varepsilon\|_{C(\mathbb{R})} \leq 1/\varepsilon$, we obtain $w_n^\varepsilon \in C^0(\mathbb{R})$, $\|w_n^\varepsilon\|_w \leq 1$,

$$0 < w_n^\varepsilon(x) \leq w_{n+1}^\varepsilon(x) \leq \frac{1}{\varepsilon + w(x)} \ \forall\, n \geq 1 \ ; \ \lim_{n\to\infty} w_n^\varepsilon(x) = \frac{1}{\varepsilon + w(x)} \ \forall x \in \mathbb{R} \ . \quad (1.7.6.4)$$

By (1.7.6.2) $\int_{\mathbb{R}} w_n^\varepsilon(x)\, d\nu(x) \leq \|L_\nu\| \ \forall n \geq 1 \ \forall \varepsilon > 0$, and according to Beppo-Levi theorem $\nu(\mathbb{R} \setminus S_w) = 0$, $1/w \in L_1(\nu)$, $\|1/w\|_{L_1(\nu)} \leq \|L_\nu\|$. That is why measure

$$\rho(A) := \int_A \frac{1}{w(x)}\, d\nu(x) \quad \forall A \in \mathcal{B}(\mathbb{R}) \quad (1.7.6.5)$$

will be positive measure in $\mathcal{M}(\mathbb{R})$, $\rho(\mathbb{R}\setminus S_w) = 0$ and $\|\rho\| \leq \|L_\nu\|$. An evident inequality $\nu(A) \leq \rho(A) \ \forall A \in \mathcal{B}(\mathbb{R})$ due to Radon-Nikodym theorem means that there exists such $\alpha \in L_1(\rho)$ that

$$\nu(A) = \int_A \alpha(x)\, d\rho(x) \quad \forall\, A \in \mathcal{B}(\mathbb{R}) \ , \quad (1.7.6.6)$$

and also $0 \leq \alpha(x) \leq 1$ almost everywhere with respect to measure ρ . Using changes of variables theorem [6, V.3], (1.7.6.5), (1.7.6.6) we get $\nu(A) = \int_A \frac{\alpha(x)}{w(x)} d\nu(x) \ \ \forall\, A \in \mathcal{B}(\mathbb{R})$, from where $\alpha(x) = w(x)$ almost everywhere with respect to measure ν , and by mutual absolute continuity of the measures ν and ρ : $\alpha(x) = w(x)$ almost everywhere with respect to measure ρ . Therefore $\nu(A) = \int_A w(x) d\rho(x) \ \ \forall\, A \in \mathcal{B}(\mathbb{R})$ and due to (1.7.6.2): $L_\nu(f) = \int_{\mathbb{R}} w(x) f(x)\, d\rho(x) \ \ \forall\, f \in C^0(\mathbb{R})$. That equality in view of density $C^0(\mathbb{R})$ in the seminormed space C_w^0 and according to the continuity of both its sides can be extended to the whole C_w^0 :

$$L_\nu(f) = \int_{\mathbb{R}} w(x) f(x)\, d\rho(x) \quad \forall\, f \in C_w^0 \ ; \quad \rho(\mathbb{R} \setminus S_w) = 0 \ . \quad (1.7.6.7)$$

Denoting constructed measures ρ by μ^+ and μ^- when ν equals to μ_L^+ and μ_L^- , respectively, and setted $\mu := \mu^+ - \mu^-$, we will get the required representation (1.6.1) taking into account $|\mu|(\mathbb{R} \setminus S_w) = 0$.

Since the inverse statement of the theorem and inequality $\|L\| \leq |\mu|(S_w)$ are evident to finish the proof one need to show only that $\|L\| \geq |\mu|(S_w)$.

Taking a Hahn expansion $\mathbb{R} = \mathbb{R}_\mu^+ \sqcup \mathbb{R}_\mu^-$ with respect to measure μ (see 1.1) and any $R > 0$ we rename introduced in (1.1.1) functions by:

$$\kappa_{n,R}^+ := \psi_n[I_R \cap \mathbb{R}_\mu^+,\ \mu] \ ; \ \ \kappa_{n,R}^- := \psi_n[I_R \cap \mathbb{R}_\mu^-,\ \mu] \ , \ n \geq 1 \ .$$

Then in view of (1.7.6.3),(1.7.6.4): $w_m^\varepsilon \cdot (\kappa_{n,R}^+ - \kappa_{n,R}^-) \in C^0(\mathbb{R})$, $\left\|w_m^\varepsilon \cdot (\kappa_{n,R}^+ - \kappa_{n,R}^-)\right\|_w \leq 1$ $\forall\, n, m \geq 1$, $R, \varepsilon > 0$, and by definition of the norm (see (1.7.6.2) and 1.1):

$$\|L\| \geq \int_{\mathbb{R}} w(x) w_m^\varepsilon(x) (\kappa_{n,R}^+(x) - \kappa_{n,R}^-(x))\, d\mu(x) \ \ \forall\, n, m \geq 1 \ , R, \varepsilon > 0 \ . \quad (1.7.6.8)$$

Passages to the limit in (1.7.6.8) as $n \to \infty$ with regard to (1.1.1) and then as $m \to \infty$ and $\varepsilon \downarrow 0$ using Beppo-Levi theorem, give us (see 1.1): $\|L\| \geq |\mu|(I_R \cap S_w) \ \ \forall\ R > 0$, i.e. $\|L\| \geq |\mu|(S_w)$. Theorem 1.3 is proved.

CHAPTER II. Hamburger criterion of the polynomial density in C_w^0 and $L_p(\mu)$, $1 \leq p < \infty$

2.1. Notations. Let $C^*(\mathbb{R})$ denote the collection of all nonnegative upper semicontinuous on the whole real line functions w satisfying condition $\|x^n w\|_{C(\mathbb{R})} < +\infty \quad \forall\ n \in \mathbb{Z}_0$ and $\mathcal{M}^*(\mathbb{R})$ – the set of all positive measures $\mu \in \mathcal{M}(\mathbb{R})$ which have all finite moments $\int_{\mathbb{R}} |x|^n \, d\mu(x) < \infty \ \ \forall n \in \mathbb{Z}_0$ and unbounded support $\operatorname{supp}\mu := \{x \in \mathbb{R} \mid \mu(x-\delta, x+\delta) > 0$ $\forall\ \delta > 0 \ \}$.

In order to abridge notations in this chapter introduce $\mathbb{R}^* := \{*\} \cup [1, +\infty)$ and for $\mu \in C^*(\mathbb{R})$ rename introduced in (1.2.1) $C_\mu^0 = \left(C_\mu^0, \|\cdot\|_\mu\right)$ by $L_*(\mu) := \left(L_*(\mu), \|\cdot\|_{L_*(\mu)}\right)$. That is why consideration $L_\alpha(\mu)$ for $1 \leq \alpha < \infty$ will mean that $\mu \in \mathcal{M}^*(\mathbb{R})$, but for $\alpha = *$ it will signify under our stipulation that $\mu \in C^*(\mathbb{R})$. For every $\alpha \in \mathbb{R}^*$ complex spaces in contrast to the real ones $L_\alpha(\mu)$ will be denoted by $L_\alpha^c(\mu)$. As well as in Chapter I: $S_\mu = \{x \in \mathbb{R} |\ \mu(x) > 0\ \}$ for $\mu \in C^*(\mathbb{R})$.

Denote for $\alpha \in \mathbb{R}^*$ and $z \in \mathbb{C}$:

$$M_n^\alpha(\mu, z) := \sup\left\{|p(z)| \ \middle|\ \|p\|_{L_\alpha(\mu)} \ \leq 1,\ p \in \mathcal{P}_n[\mathbb{C}]\ \right\}\ ,\ n \in\ \mathbb{Z}_0\ , \tag{2.1.1}$$

$$\rho_n^\alpha(\mu, z) := \inf\left\{\|p\|_{L_\alpha(\mu)} \ \middle|\ |p(z)| \ = 1,\ p \in \mathcal{P}_n[\mathbb{R}]\ \right\}\ ,\ n \in\ \mathbb{Z}_0\ , \tag{2.1.2}$$

$$M_\alpha(\mu, z) := \lim_{n\to\infty} M_n^\alpha(\mu, z)\ ; \quad \rho_\alpha(\mu, z) := \lim_{n\to\infty} \rho_n^\alpha(\mu, z)\ . \tag{2.1.3}$$

It easy to verify that

$$\frac{1}{\rho_n^\alpha(\mu, z)} = \sup\left\{|p(z)| \ \middle|\ \|p\|_{L_\alpha(\mu)} \leq 1, p \in \mathcal{P}_n[\mathbb{R}]\right\}, \frac{1}{\rho_n^\alpha(\mu, z)} \leq M_n^\alpha(\mu, z) \leq \frac{2}{\rho_n^\alpha(\mu, z)}. \tag{2.1.4}$$

Introduce

$$\left\{\begin{array}{llll} d\mu_\alpha(x) := \frac{1}{(1+|x|)^\alpha} d\mu(x)\ ; & d\mu_\alpha^{(2)}(x) := \frac{|x|^\alpha}{(1+|x|)^\alpha} d\mu(x)\ ; & 1 \leq \alpha < \infty,\ \mu \in \mathcal{M}^*(\mathbb{R})\ ; \\ \mu_*(x) := \frac{1}{1+|x|}\mu(x)\ ; & \mu_*^{(2)}(x) := \frac{|x|}{1+|x|}\mu(x)\ ; & \mu \in C^*(\mathbb{R})\ . \end{array}\right. \tag{2.1.5}$$

Restricting the polynomial class in (2.1.1) and (2.1.4) to the vanishing at zero polynomials we get for $z \in \mathbb{C}$, $n \geq 1$ and $\alpha \in \mathbb{R}^*$:

$$\left\{\begin{array}{ll} M_n^\alpha(\mu_\alpha, z) \ \geq \ |z|\ M_{n-1}^\alpha(\mu_\alpha^{(2)}, z)\ ; & \rho_{n-1}^\alpha(\mu_\alpha^{(2)}, z) \ \geq \ |z|\ \rho_n^\alpha(\mu_\alpha, z)\ ; \\ M_\alpha(\mu_\alpha, z) \ \geq \ |z|\ M_\alpha(\mu_\alpha^{(2)}, z)\ ; & \rho_\alpha(\mu_\alpha^{(2)}, z) \ \geq \ |z|\ \rho_\alpha(\mu_\alpha, z)\ . \end{array}\right. \tag{2.1.6}$$

2.2. Background. Functions $\rho_n(z) := \frac{1}{M_n^2(\mu,z)}$, $n \in \mathbb{Z}_0$, were introduced by H. Hamburger [15] in connection with the investigation of an indeterminate moment

problem. These functions were used by M. Riesz [35] to obtain the criterion of the polynomial density in $L_2(\mu)$. For $\alpha = *$ and discrete set S_μ function $M_*(\mu, z)$ was introduced by T. Holl [19] and for an arbitrary $\mu \in C^*(\mathbb{R})$ - by S. Mergelyan in [32]. B.Ja.Levin [30] generalized these results in the following statement a simpler proof of which was found recently by Ch. Berg [8].

Proposition 2.1.([30]) *Let $\alpha \in \mathbb{R}^*$. If $\mathcal{P}[\mathbb{C}]$ is dense in $L^c_\alpha(\mu)$ then $M_\alpha(\mu_\alpha; z) = \infty \;\; \forall\, z \in \mathbb{C} \setminus \mathrm{supp}\mu$. If $\exists z \in \mathbb{C} \setminus \mathrm{supp}\mu$: $M_\alpha(\mu_\alpha; z) = \infty$, then $\mathcal{P}[\mathbb{C}]$ is dense in $L^c_\alpha(\mu)$.*

Denote by $\mathrm{Close}_{L_\alpha(\mu)} A$ the closure of $A \subseteq L_\alpha(\mu)$ in the space $L_\alpha(\mu)$. It is known and it can be easily seen from the Proposition 2.1 and (2.1.4) that $\mathcal{P}[\mathbb{C}]$ is dense in $L^c_\alpha(\mu)$ if and only if $\mathcal{P}[\mathbb{R}]$ is dense in $L_\alpha(\mu)$. That is why everywhere below we will examine only real case and use the following statement.

Proposition 2.2. ([30; 2, Th.2.3.2]) *Let $\alpha \in \mathbb{R}^*$. The following statements are equivalent:*

(2.2.1a) $\mathrm{Close}_{L_\alpha(\mu)}\mathcal{P} = L_\alpha(\mu)$; (2.2.1$d$) $\exists z \in \mathbb{C} \setminus \mathrm{supp}\mu \; : \rho_\alpha(\mu_\alpha, z) = 0$;
(2.2.1b) $\mathrm{Close}_{L^c_\alpha(\mu)}\mathcal{P}[\mathbb{C}] = L^c_\alpha(\mu)$; (2.2.1$e$) $\rho_\alpha(\mu_\alpha, z) = 0 \;\; \forall z \in \mathbb{C} \setminus \mathrm{supp}\mu$;
(2.2.1c) $\frac{1}{x+i} \in \mathrm{Close}_{L^c_\alpha(\mu)}\mathcal{P}[\mathbb{C}]$; (2.2.1g) $\frac{1}{1+x^2}, \frac{x}{1+x^2} \in \mathrm{Close}_{L_\alpha(\mu)}\mathcal{P}$.

H.Hamburger in [15] established another criterion of the indeterminacy of a moment problem a simpler proof of which was given by M. Riesz [36]. This criterion can be formulated as follows:

$$\mathrm{Close}_{L_2(\mu)}\mathcal{P} \neq L_2(\mu) \;\Leftrightarrow\; \rho_2(\mu_2, 0) > 0 \text{ and } \rho_2(\mu_2^{(2)}, 0) > 0 \; . \tag{2.2.2}$$

Succeeding Berg's proof [8] of the Proposition 2.1 and using Theorem 1.3 we will extend here criterion (2.2.2) to all spaces $L_\alpha(\mu)$, $\alpha \in \mathbb{R}^*$, designated

$$\rho^\alpha_n(\mu) := \rho^\alpha_n(\mu, 0) \; , \quad n \in \mathbb{Z}_0 \; ; \qquad \rho_\alpha(\mu) := \rho_\alpha(\mu, 0) \; , \quad \alpha \in \mathbb{R}^* \; . \tag{2.2.3}$$

2.3. Main Theorem.

Arbitrary change of zeros of some polynomial $p \in \mathcal{P}[\mathbb{C}]$ by the complex conjugate ones gives the polynomial set $\pi(p)$ containing only one polynomial $p^* \in \pi(p)$ all zeros of which lie in the lower complex halfplane $\mathbb{C}^- := \{z \in \mathbb{C} \mid \mathrm{Im}z \le 0\}$. It is evident that $|q(x)| = |p(x)| \;\; \forall x \in \mathbb{R} \; \forall \; q \in \pi(p)$, and therefore for any $\alpha \in \mathbb{R}^*$: $\|q\|_{L_\alpha(\mu)} = \|p\|_{L_\alpha(\mu)}$ $\forall \; q \in \pi(p)$. Besides that for arbitrary $a \in \mathbb{R}$ and $y \ge 0$: $|p^*(a+iy)| \ge |q(a+iy)|$ $\forall \; q \in \pi(p)$, and $|p^*(a+iy)|$ is a nondecreasing function of $y \ge 0$. That is why for any $\alpha \in \mathbb{R}^*$, $n \in \mathbb{Z}_0$, $a \in \mathbb{R}$ and $y \ge 0$:

$$M^\alpha_n(\mu, a+iy) = \sup\left\{ |p^*(a+iy)| \;\middle|\; \|p\|_{L^c_\alpha(\mu)} \le 1 \; , \; p \in \mathcal{P}_n[\mathbb{C}] \right\} \; , \tag{2.3.1}$$

and $M^\alpha_n(\mu, a+iy)$ is a nondecreasing function of $y \ge 0$. Thus, an obvious equality $M^\alpha_n(\mu, z) = M^\alpha_n(\mu, \overline{z}) \; \forall z \in \mathbb{C}$ implies validity of the following statement.

PROPOSITION 2.3. *For arbitrary $\alpha \in \mathbb{R}^*$, $a \in \mathbb{R}$ and $n \in \mathbb{Z}_0$ functions $M_n^\alpha(\mu, a+iy)$, $M_\alpha(\mu, a+iy)$ of the variable $y \in \mathbb{R}$ are even on $\mathbb{R}$ and nondecreasing on $[0, +\infty)$ and, in particular,*

$$\frac{1}{\rho_\alpha(\mu, x)} \leq M_\alpha(\mu, x) \leq M_\alpha(\mu, x+iy) \quad \forall \;\; x, y \in \mathbb{R}, \alpha \in \mathbb{R}^* \ , \tag{2.3.2}$$

where $\frac{1}{0} := +\infty$ and $+\infty \leq +\infty$.

The following criterion of the polynomial density in $L_\alpha(\mu)$, $\alpha \in \mathbb{R}^*$, is the main result of that Chapter.

THEOREM 2.1. (*Hamburger local criterion*) *Let $\alpha \in \mathbb{R}^*$. Algebraic polynomials $\mathcal{P}$ are not dense in $L_\alpha(\mu)$ if and only if*

$$\rho_\alpha(\mu_\alpha) > 0 \quad \text{and} \quad \rho_\alpha(\mu_\alpha^{(2)}) > 0 \ , \tag{2.3.3}$$

where μ_α , $\mu_\alpha^{(2)}$ are defined in (2.1.5) and (see 2.1)

$$\rho_\alpha(\nu) = \inf \Big\{ \|p\|_{L_\alpha(\nu)} \mid p(0) = 1, \ p \in \mathcal{P} \Big\} \ , \quad \nu \in \{ \mu_\alpha, \ \mu_\alpha^{(2)} \ \} \ , \tag{2.3.4}$$

2.4. Proof of Theorem 2.1.

2.4.1. *Sufficiency.* Let $\alpha \in \mathbb{R}^*$ and (2.3.3) is valid. Then $\mu_\alpha^{(2)} \not\equiv 0$ and consequently, $S_\mu \setminus \{0\} \neq \emptyset$, if $\alpha = *$, and supp$\mu \setminus \{0\} \neq \emptyset$, if $\alpha \in [1, +\infty)$. Due to (2.1.4)

$$|p(0)| \leq \frac{\|p\|_{L_\alpha(\mu_\alpha)}}{\rho_\alpha(\mu_\alpha)} \ ; \quad |p(0)| \leq \frac{\|p\|_{L_\alpha(\mu_\alpha^{(2)})}}{\rho_\alpha(\mu_\alpha^{(2)})} \quad \forall \ p \in \mathcal{P} \ . \tag{2.4.1}$$

By Hahn-Banach theorem, (2.4.1) and Theorem 1.3 there exist such $f_\alpha \in L_\beta(\mu_\alpha) \setminus \{0\}$, $g_\alpha \in L_\beta(\mu_\alpha^{(2)}) \setminus \{0\}$, if $\alpha \in [1, +\infty)$ where α , β are dual exponents, and such $\kappa, \gamma \in \mathcal{M}(\mathbb{R})$, $|\kappa|(S_\mu) > 0$, $|\kappa|(\mathbb{R} \setminus S_\mu) = 0$, $|\gamma|(S_\mu \setminus \{0\}) > 0$, $|\gamma|(\mathbb{R} \setminus (\ S_\mu \setminus \{0\}) \ ) = 0$, if $\alpha = *$, that respectively to the considered cases:

$$p(0) = \int_{\mathbb{R}} p(t) f_\alpha(t) \ d\mu_\alpha(t) \ = \ \int_{\mathbb{R}} p(t) g_\alpha(t) \ d\mu_\alpha^{(2)}(t) \quad \forall \ p \in \mathcal{P} \ , \tag{2.4.2}$$

$$p(0) = \int_{\mathbb{R}} \mu_*(t) p(t) \ d\kappa(t) \ = \ \int_{\mathbb{R}} \mu_*^{(2)}(t) p(t) \ d\gamma(t) \quad \forall \ p \in \mathcal{P} \ . \tag{2.4.3}$$

Consider at first the case $\alpha = *$. According to the notations (2.1.5) equality (2.4.3) can be rewritten in the following way:

$$p(0) = \int_{\mathbb{R}} \mu(t) p(t) \ d\kappa_1(t) \ = \ \int_{\mathbb{R}} \mu(t) p(t) \ d\gamma_1^{(2)}(t) \quad \forall \ p \in \mathcal{P} \ , \tag{2.4.4}$$

where, obviously, $\kappa_1, \gamma_1^{(2)} \in \mathcal{M}(\mathbb{R})$. Applying left equality (2.4.4) to the polynomials vanishing at zero we get:

$$0 = \int_{\mathbb{R}} \mu(t)p(t)\ d\,\widetilde{\kappa}(t) \quad \forall\, p \in \mathcal{P}\ , \quad d\,\widetilde{\kappa}(t) := \frac{t}{1+|t|} d\kappa(t)\ .$$

If $|\kappa|(S_\mu \setminus \{0\}) > 0$ then by Theorem 1.3 $\widetilde{\kappa} \in L_*(\mu)^* \setminus \{0\}$ and therefore $\mathrm{Close}_{L_*(\mu)}\mathcal{P} \neq L_*(\mu)$. If $|\kappa|(S_\mu \setminus \{0\}) = 0$, then $0 \in S_\mu$ and $|\kappa|(\mathbb{R} \setminus \{0\}) = 0$ but by (2.4.3) $d\kappa = (1/\mu(0))\delta_0$, where δ_0 - Dirac's measure [13, 4.4.1] at the point 0 . That is why it follows from (2.4.4) that

$$0 = \int_{\mathbb{R}} \mu(t)p(t)\ d(\kappa_1(t) - \gamma_1^{(2)}(t)) \quad \forall\, p \in \mathcal{P}\ ,$$

and in addition $(\kappa_1 - \gamma_1^{(2)})(\{0\}) = 1/\mu(0)$. This means due to Theorem 1.3 and $0 \in S_\mu$ that $\kappa_1 - \gamma_1^{(2)} \in L_*(\mu)^* \setminus \{0\}$ and hence, $\mathrm{Close}_{L_*(\mu)}\mathcal{P} \neq L_*(\mu)$.

Let now $\alpha \in [1, +\infty)$. Applying left equality (2.4.2) to the vanishing at zero polynomials we get

$$\int_{\mathbb{R}} p(t)\frac{tf_\alpha(t)}{(1+|t|)^\alpha}\ d\mu(t)\ =\ 0\ \ \forall\, p \in \mathcal{P}\ .$$

If (see 1.1) $\mu(S_{|f_\alpha|} \setminus \{0\}) > 0$ then $\frac{tf_\alpha(t)}{(1+|t|)^\alpha} \in L_\beta(\mu) \setminus \{0\}$ and consequently, $\mathrm{Close}_{L_\alpha(\mu)}\mathcal{P} \neq L_\alpha(\mu)$. But if $\mu(S_{|f_\alpha|} \setminus \{0\}) = 0$ then $\mu(\{0\}) > 0$, $0 \in S_{|f_\alpha|}$ and by (2.4.2): $\mu(\{0\}) = 1/f_\alpha(0) > 0$. Equalities (2.4.2) yield:

$$0 = \int_{\mathbb{R}} p(t)\varphi_\alpha(t)\ d\mu(t)\ \ \forall\, p \in \mathcal{P}\ , \quad \varphi_\alpha(t) = \frac{f_\alpha(t) - |t|^\alpha g_\alpha(t)}{(1+|t|)^\alpha}\ .$$

It is easy to verify that $\varphi_\alpha \in L_\beta(\mu)$ and if $d\mu_0 := d\mu - \frac{1}{f_\alpha(0)} \cdot \delta_0$ then for arbitrary $\varepsilon > 0$:

$$\|\varphi_\alpha\|^\beta_{L_\beta(\mu)}\ \geq\ \int\limits_{-\varepsilon}^{+\varepsilon} |\varphi_\alpha(t)|^\beta\ d\mu_0(t)\ +\ \frac{1}{f_\alpha(0)}|\varphi_\alpha(0)|^\beta\ \geq\ f_\alpha(0)^{\beta-1}\ >\ 0\ ,$$

if $\alpha > 1$, and $\|\varphi_1\|_{L_\infty(\mu)} \geq f_\alpha(0) > 0$, if $\alpha = 1$, i.e. $\varphi_\alpha \in L_\beta(\mu) \setminus \{0\}$ and hence, $\mathrm{Close}_{L_\alpha(\mu)}\mathcal{P} \neq L_\alpha(\mu)$.

2.4.2. *Necessity.* Let $\alpha \in \mathbb{R}^*$ and $\mathrm{Close}_{L_\alpha(\mu)}\mathcal{P} \neq L_\alpha(\mu)$. Then by Hahn-Banach theorem and Theorem 1.3 for $\alpha \in [1, +\infty)$ there exists such $g_\alpha \in L_\beta(\mu) \setminus \{0\}$, $1/\alpha + 1/\beta = 1$, that:

$$\int\limits_{\mathbb{R}} p(t)g_\alpha(t)\ d\mu(t)\ =\ 0\ \ \forall\, p \in \mathcal{P}\ , \tag{2.4.5}$$

and if $\alpha = *$ then $\exists \gamma \in \mathcal{M}(\mathbb{R})$, $|\gamma|(S_\mu) > 0$, $|\gamma|(\mathbb{R} \setminus S_\mu) = 0$:

$$\int\limits_{\mathbb{R}} \mu(t)p(t)\ d\gamma(t)\ =\ 0\ \ \forall\, p \in \mathcal{P}\ . \tag{2.4.6}$$

Under these conditions function

$$\varphi_\alpha(z) := \int\limits_{\mathbb{R}} \frac{g_\alpha(t)}{t-z}\ d\mu(t),\ \text{if } 1 \le \alpha < +\infty\ ;\ \varphi_*(z) := \int\limits_{\mathbb{R}} \frac{\mu(t)}{t-z}\ d\gamma(t),\ \text{if } \alpha = *\ ; \quad (2.4.7)$$

is analytic on $\mathbb{C}\setminus\mathbb{R}$ and not identically zero. Thus, $\exists \lambda_\alpha \in [1,2] : \varphi_\alpha(i\lambda_\alpha) \neq 0$. Besides that it is easy to derive from (2.4.5) and (2.4.6) that for any $z \in \mathbb{C}\setminus\mathbb{R}$ and $p \in \mathcal{P}[\mathbb{C}]$:

$$p(z)\varphi_\alpha(z) := \int\limits_{\mathbb{R}} \frac{p(t)g_\alpha(t)}{t-z}\ d\mu(t), 1 \le \alpha < +\infty\ ;\ p(z)\varphi_*(z) := \int\limits_{\mathbb{R}} \frac{p(t)\mu(t)}{t-z}\ d\gamma(t), \alpha = *\ ; \quad (2.4.8)$$

Setted in (2.4.8) $z = i\lambda_\alpha$ we get for $\alpha \in [1,+\infty)$:

$$|p(i\lambda_\alpha)| \le \frac{\|g_\alpha\|_{L_\beta(\mu)}}{|\varphi_\alpha(i\lambda_\alpha)|}\left[\int\limits_{\mathbb{R}} \frac{|p(t)|^\alpha}{|t-i\lambda_\alpha|^\alpha}\ d\mu(t)\right]^{1/\alpha} \le \frac{\sqrt{2}\,\|g_\alpha\|_{L_\beta(\mu)}}{|\varphi_\alpha(i\lambda_\alpha)|}\|p\|_{L_\alpha(\mu_\alpha)}\ , \quad (2.4.9)$$

and for $\alpha = *$:

$$|p(i\lambda_\alpha)| \le \frac{\|p\|_{L_*(\mu_*)}}{|\varphi_*(i\lambda_*)|}\int\limits_{\mathbb{R}} \frac{1+|t|}{|t-i\lambda_*|}\ d|\gamma|(t) \le \frac{\sqrt{2}\,\|\gamma\|}{|\varphi_*(i\lambda_*)|}\|p\|_{L_*(\mu_*)}\ . \quad (2.4.10)$$

Thus, $M_\alpha(\mu_\alpha, i\lambda_\alpha) < \infty$ and by (2.3.2): $1/\rho_\alpha(\mu_\alpha) \le M_\alpha(\mu_\alpha, i\lambda_\alpha) < \infty$, what together with arised from (2.3.2) and (2.1.6) inequality

$$\frac{1}{\rho_\alpha(\mu_\alpha^{(2)})} \le M_\alpha(\mu_\alpha^{(2)}, i\lambda_\alpha) \le \frac{1}{|i\lambda_\alpha|}M_\alpha(\mu_\alpha, i\lambda_\alpha)\ <\ \infty\ ,$$

gives validity of the inequalities (2.2.2). Theorem 2.1 is proved.

CHAPTER III. Hamburger and Krein classes of entire functions

3.1. Notations and Definitions. A function $f : \mathbb{C} \to \mathbb{C}$ is said to be of *exponential type* if $|f(z)| \le Ce^{\sigma|z|}$ $\forall z \in \mathbb{C}$ for some $\sigma, C > 0$, and of *minimal exponential* type if

$$\forall \varepsilon > 0\ \ \exists\ C_\varepsilon > 0\ :\ \ |f(z)| \le C_\varepsilon e^{\varepsilon|z|}\quad \forall\ z \in \mathbb{C}\ . \quad (3.1.1)$$

Let $\mathcal{E}, \mathcal{E}_1, \mathcal{E}_0$ denote the sets of all entire functions, entire functions of exponential type and entire functions of minimal exponential type, respectively; Λ_f – the set of all zeros $f \in \mathcal{E}$; $\mathrm{Close}_{\mathcal{E}}A$ – the closure $A \subseteq \mathcal{E}$ with respect to topology $\tau_{\mathcal{E}}$ of the uniform convergence on all compact subsets of $\mathbb{C}$; $\mathrm{Close}_{L_\alpha(\mu)}A$ – the closure $A \subseteq L_\alpha(\mu)$ in the space $L_\alpha(\mu)$, $\alpha \in \mathbb{R}^*$ (see 2.1); $\deg P$ – degree of the polynomial $P \in \mathcal{P}[\mathbb{C}]$; $a \vee b := \max\{a, b\}$, $a, b \in \mathbb{R}$; $\mathrm{co}A$ – convex hull of $A \subseteq \mathbb{C}$; $\mathrm{card}B \in \mathbb{Z}_0 \cup \{\infty\}$ – number of elemens in the set B . Function $f \in \mathcal{E}$ is said

to be real if $f(\mathbb{R}) \subseteq \mathbb{R}$. For $n \in \mathbb{Z}_0$ and $X \in \{\mathcal{P}_n, \mathcal{P}, \mathcal{E}, \mathcal{E}_1, \mathcal{E}_0\}$ let $X(\mathbb{R})$ denote the set of real functions from X with real zeros only and $X^*(\mathbb{R})$ – the set of real functions $f \in X$ all zeros of which are real, simple and $f(0) = 1$. The sets of real functions from $\mathcal{P}, \mathcal{E}$ with only real and simple zeros will be denoted by $\mathcal{P}_s(\mathbb{R}), \mathcal{E}_s(\mathbb{R})$, respectively.

Let us remind that [30, VIII.1] the set $A \subseteq \mathcal{E}$ is said to be *normal* if $\mathrm{Close}_{\mathcal{E}} A$ is a compact set with respect to the topology $\tau_{\mathcal{E}}$ or, what is the same [26, IV.41, II.20; 37, I.6.1], if any sequence in A contains the convergent subsequence with respect to the same topology $\tau_{\mathcal{E}}$.

3.1.1. Cartwright class. The set of entire functions $f \in \mathcal{E}_1$ satisfying inequality

$$\int_{\mathbb{R}} \frac{\log^+ |f(t)|}{1+t^2}\, dt \ < \infty \ , \quad \log^+ x := 0 \vee \log x, \ x \geq 0 \ , \tag{3.1.2}$$

is called the Cartwright class and will be denoted by Cartwright . Each $f \in$ Cartwright is an element of so-called (A) class of entire functions, i.e. $\sum_{\lambda \in \Lambda_f \setminus \{0\}} \left|\mathrm{Im}\frac{1}{\lambda}\right| < \infty$, and satisfies [29, V.4, Th.7] the stronger inequality:

$$\int_{\mathbb{R}} \frac{|\log |f(t)||}{1+t^2}\, dt \ < \infty \quad . \tag{3.1.3}$$

It is known also [29, V.6, Th.13] that $f \in \mathrm{Cartwright} \cap \mathcal{E}(\mathbb{R})$ has the following representation:

$$f(z) = c \cdot z^m \lim_{R \to +\infty} \prod_{\lambda \in (-R,R) \cap \left(\Lambda_f \setminus \{0\}\right)} \left(1 - \frac{z}{\lambda}\right) \ , \ z \in \mathbb{C}, \ c \in \mathbb{R} \setminus \{0\}\, , \ m \in \mathbb{Z}_0 \ , \tag{3.1.4}$$

and there exist the finite limits: $\lim\limits_{R \to +\infty} \frac{\mathrm{card}\left(\Lambda_f \cap [0,R]\right)}{R} = \lim\limits_{R \to +\infty} \frac{\mathrm{card}\left(\Lambda_f \cap [-R,0]\right)}{R}$,

$$\delta_f := \lim_{R \to +\infty} \delta_f(R) \ ; \quad \delta_f(R) := \sum_{\lambda \in (-R,R) \cap \left(\Lambda_f \setminus \{0\}\right)} \frac{1}{\lambda} \ . \tag{3.1.5}$$

It is worth to remind that Lindelof and Hadamard's [29, I] theorems for any $f \in \mathcal{E}_0(\mathbb{R})$ give an existence of the finite limit (3.1.5), equality $\delta_f = 0$ and validity of the representation (3.1.4).

3.1.2. Krein class. According to [2, III] and [25] (see also [8, 9, 7, 39]) function $f \in \mathcal{E}_s(\mathbb{R})$ is said to be a function of Krein class $\mathcal{K}$ if its reciprocal can be represented as a series of simple fractions:

$$\frac{1}{f(z)} \ = \ A + \frac{B}{z} + \sum_{n \geq 1} A_n \ \frac{z}{\lambda_n (z - \lambda_n)} \ , \ z \in \mathbb{C} \setminus \{\lambda_n\}_{n \geq 1} \ ;$$

where $\lambda_n \neq 0, A, B, A_n, \lambda_n \in \mathbb{R}\ \forall\ n \geq 1$ and $\sum\limits_{n \geq 1} \frac{|A_n|}{{\lambda_n}^2} \ < \ \infty$.

3.1.3. Related to the Krein class definitions. For every $f \in \mathcal{E}_s(\mathbb{R})$ define

$$d_f \ := \ \inf \ \left\{ q \in \mathbb{Z} \mid \sum_{\lambda \in \Lambda_f \setminus \{0\}} \frac{1}{|f'(\lambda)| \cdot |\lambda|^{q+1}} \ < \ \infty \right\} \ , \tag{3.1.6}$$

considering $\inf\emptyset := +\infty$. If $f \in \mathcal{E}_s(\mathbb{R})$ and $d_f < +\infty$, then for every $p \in \mathbb{Z}_0$, $p \geq d_f$ one can introduce entire function:

$$\Delta_f^p(z) := \frac{1}{f(z)} - \frac{\chi_{\Lambda_f}(0)}{f'(0)\cdot z} - \sum_{\lambda\in\Lambda_f\setminus\{0\}} \frac{z^p}{\lambda^p f'(\lambda)(z-\lambda)}\ , z \in \mathbb{C}\ , \tag{3.1.7}$$

and meromorphic function:

$$m_f^p(z) := \frac{\chi_{\Lambda_f}(0)}{f'(0)\cdot z} + \sum_{k=0}^{p-1} \frac{z^k}{k!}\left(\frac{1}{f(z)} - \frac{\chi_{\Lambda_f}(0)}{f'(0)\cdot z}\right)^{(k)}(0) + \sum_{\lambda\in\Lambda_f\setminus\{0\}} \frac{z^p}{\lambda^p f'(\lambda)(z-\lambda)}\ , z \in \mathbb{C}\ , \tag{3.1.8}$$

where $\sum\limits_{k=0}^{-1} := 0$.

3.1.4. Hamburger class. Hamburger in [16] defined the class $\mathcal{H}$ of entire functions $f \in \mathcal{E}_s(\mathbb{R})$, satisfying the following two conditions:

(3.1.9a) $\frac{1}{f(z)} = \sum\limits_{\lambda\in\Lambda_f} \frac{1}{f'(\lambda)(z-\lambda)}\ ,\ z \in \mathbb{C}\setminus\Lambda_f$;

(3.1.9b) $\sum\limits_{\lambda\in\Lambda_f} \frac{|\lambda|^n}{|f'(\lambda)|} < \infty\ \ \forall\, n \in \mathbb{Z}_0$.

3.1.5. Laguerre-Pólya class. We will consider below a certain subclass of the well-known second Laguerre-Pólya class [21, p.336; 18, III.3, Def.3.1; 29, VIII; 27]:

$$\mathcal{LP}_{II} := \left\{ b e^{az-c^2z^2} \prod_{n\geq 1}(1-\frac{z}{\lambda_n})e^{\frac{z}{\lambda_n}} \mid a, c \in \mathbb{R},\ b, \lambda_n \in \mathbb{R}\setminus\{0\}\ \forall n \geq 1,\ \sum_{n\geq 1}\frac{1}{\lambda_n^2} < \infty \right\},$$

namely,

$$\mathcal{LP}_{II}^0 := \left\{ b e^{az} \prod_{n\geq 1}(1-\frac{z}{\lambda_n})e^{\frac{z}{\lambda_n}} \mid a \in \mathbb{R},\ b, \lambda_n \in \mathbb{R}\setminus\{0\}\ \forall n \geq 1,\ \sum_{n\geq 1}\frac{1}{\lambda_n^2} < \infty \right\}. \tag{3.1.10}$$

3.2. Normal polynomial families in $L_p(\mu)$.

3.2.1. It follows from the definition of $M_\alpha(\mu, z)$ that for every $\alpha \in \mathbb{R}^*$ the following implication holds:

$$p \in \mathcal{P}[\mathbb{C}]\ ,\ \ \|p\|_{L_\alpha^c(\mu)} \leq 1 \ \Rightarrow\ |p(z)| \leq M_\alpha(\mu, z)\ \forall\ z \in \mathbb{C}\ . \tag{3.2.1}$$

Inequality (3.2.1) become essential when $\mathrm{Close}_{L_\alpha(\mu)}\mathcal{P} \neq L_\alpha(\mu)$. It is known that in this case (2.4.8), (2.4.9) and (2.4.10) imply the uniform boundedness of $M_\alpha(\mu, z)$ on any segment of the form $ia + b\cdot I$, $a \in \mathbb{R}\setminus\{0\}$, $b > 0$, which also does not include any zero of the defined in (2.4.7) function $\varphi_\alpha(z)$. In view of the Proposition 2.3 this means the uniform boundedness of $M_\alpha(\mu, z)$ on any compact subset of the complex plane $\mathbb{C}$. Thus, by virtue of Vitali's classical compactness theorem and (3.2.1) we deduce that each subset

$$\mathcal{P}_\alpha(\mu) := \Big\{ p \in \mathcal{P} \mid \|p\|_{L_\alpha(\mu)} \le 1 \Big\} \;,\; \mathcal{P}^c_\alpha(\mu) := \Big\{ p \in \mathcal{P}[\mathbb{C}] \mid \|p\|_{L^c_\alpha(\mu)} \le 1 \Big\} \;, \tag{3.2.2}$$

is normal and so their closures $\mathcal{E}_\alpha(\mu) := \mathrm{Close}_\mathcal{E}\mathcal{P}_\alpha(\mu)$, $\mathcal{E}^c_\alpha(\mu) := \mathrm{Close}_\mathcal{E}\mathcal{P}^c_\alpha(\mu)$ are compact sets in the topology $\tau_\mathcal{E}$ and moreover

$$|f(z)| \le M_\alpha(\mu, z) \;\forall\; z \in \mathbb{C} \;\;\forall\; f \in \mathcal{E}^c_\alpha(\mu) \;. \tag{3.2.3}$$

Inequality (3.2.1) for $\alpha = 2$ was indicated in [38, Th.2.5] where also the problem about the complete description of the set $\mathcal{E}^c_\alpha(\mu)$ was raised. In addition, known M. Riesz's theorems [2, Th.2.4.1,2.4.3] assert that the function $M_\alpha(\mu, z)$ for $\alpha = 2$ is of minimal exponential type and inequality (3.1.2) holds for $f(t) = M_\alpha(\mu, t)$ there. These two properties of $M_\alpha(\mu, z)$ were proved for $\alpha = *$ by S. Mergelyan in [32] and for arbitrary $\alpha \in \mathbb{R}^*$ - by B.Ja.Levin in [30].

Observe, that for arbitrary $\alpha \in \mathbb{R}^*$ condition (3.1.2) for $f(t) = M_\alpha(\mu, t)$ in view of the evident lower bound:

$$M_\alpha(\mu, z) \;\ge\; \frac{1}{\|\mu\|_\alpha} \quad \forall\; z \in \mathbb{C} \;,\;\; \|\mu\|_\alpha := \begin{cases} |\mu|(\mathbb{R}) \;, & 1 \le \alpha < \infty; \\ \|\mu\|_{C(\mathbb{R})} \;, & \alpha = *; \end{cases} \tag{3.2.4}$$

is equivalent to the condition (3.1.3) with the same $f(t)$. In the next item 3.2.2 we will prove the following statement using proofs from [32, item 13] and [2, Th.2.4.3].

Proposition 3.1. ([30, 32, 36, 2]) *Let* $\alpha \in \mathbb{R}^*$ *and* $\mathrm{Close}_{L_\alpha(\mu)}\mathcal{P} \ne L_\alpha(\mu)$. *Then function* $M_\alpha(\mu, \cdot) : \mathbb{C} \to (0, +\infty)$ *is of minimal exponential type and inequality (3.1.3) holds for* $f(t) = M_\alpha(\mu, t)$.

That is why Proposition 3.1 together with inequality (3.2.3) shows that for arbitrary $\alpha \in \mathbb{R}^*$ incompleteness of $\mathcal{P}$ in $L_\alpha(\mu)$ means that

$$\mathcal{E}^c_\alpha(\mu),\; \mathcal{E}_\alpha(\mu) \subseteq\; \mathcal{E}_0 \cap \mathrm{Cartwright} \tag{3.2.5}$$

and therefore in that case polynomials can approximate in the space $L_\alpha(\mu)$ only those functions which from their domain of definition in $L_\alpha(\mu)$ can be extended into the whole complex plane as an entire function of minimal exponential type from the Cartwright class.

3.2.2. Proof of Proposition 3.1.

Let $\alpha \in \mathbb{R}^*$. Since $M_\alpha(\lambda \cdot \mu, z) = \frac{1}{\lambda} M_\alpha(\mu, z) \;\forall \lambda > 0$, then without loss of generality one may consider $\|\mu\|_\alpha = 1$. By Proposition 2.3 and (3.2.4) to prove inequality (3.1.1) it is sufficient to show that:

$$\forall \varepsilon > 0 \;\exists C_\varepsilon > 0 : \;\log|M_\alpha(\mu, z)| \le\; C_\varepsilon + \varepsilon|y| \;\;\forall\; z = x + iy \in \mathcal{A} \;, \tag{3.2.6}$$

where $\mathcal{A} := \{ z \in \mathbb{C} \mid y \ge 2 + |x| \;\}$. But defined in (2.4.7) function $\varphi_\alpha(z)$ is uniformly bounded in $\mathrm{Im} z \ge 1$ and so by known corollary of Jensen theorem [22, IV.D, VI.C]:

$$\int\limits_\mathbb{R} \frac{|\; \log|\psi(t)|\; |}{1 + t^2}\, dt \;\; < \infty \quad , \tag{3.2.7}$$

for $\psi(t) = \varphi_\alpha(1+it)$, $t \in \mathbb{R}$. Using the similar to (2.4.9) and (2.4.10) estimates one can easily obtain an existence of such constant $C_\alpha > 0$ that:

$$|p(1+it)| \leq \Phi(t) \quad \forall\, t \in \mathbb{R} \quad \forall\, p \in \mathcal{P}^c_\alpha(\mu)\ , \tag{3.2.8}$$

where:

$$\Phi(t) := \frac{C_\alpha}{|\varphi(1+it)|} \quad \forall\, t \in \mathbb{R}\ . \tag{3.2.9}$$

Since (3.2.7) is valid and for $\psi = \Phi$, then using the Poisson formula [29, V.2, Th.4] for the harmonic in $\mathrm{Im} z \geq 1$ function $\log p^*(z)$ (polynomial $p^* \in \pi(p) \subseteq \mathcal{P}^c_\alpha(\mu)$ has been defined at the beginning of the section 2.3.) we will get from the formulas (2.1.4), (2.3.1) for $z = x + iy \in \mathcal{A}$ the following inequalities:

$$\log|M_\alpha(\mu,z)| = \sup_{p\in\mathcal{P}^c_\alpha(\mu)} \log|p^*(z)| = \sup_{p\in\mathcal{P}^c_\alpha(\mu)} \frac{y-1}{\pi}\int\limits_{\mathbb{R}} \frac{\log|p^*(i+t)|}{(t-x)^2+(y-1)^2}\,dt \leq$$

$$\leq \frac{y-1}{\pi}\int\limits_{\mathbb{R}} \frac{\gamma(t)}{(t-x)^2+(y-1)^2}\,dt\ ,$$

where $\gamma(t) := |\log\Phi(t)|$, $t \in \mathbb{R}$. Since for all $z \in \mathcal{A}$: $(t-x)^2 + (y-1)^2 \geq \frac{1}{2}(1+t^2)$ $\forall\, t \in \mathbb{R}$, then validity for arbitrary $T > 0$ and $z \in \mathcal{A}$ of the following relations (cf. [2, Th.2.4.3, Proof]):

$$\int\limits_{|t|\geq T} \frac{\gamma(t)}{(t-x)^2+(y-1)^2}\,dt \leq\ 2\int\limits_{|t|\geq T}\frac{\gamma(t)}{1+t^2}\,dt;\ \lim_{y\to+\infty}\int\limits_{-T}^{T}\frac{\gamma(t)}{(t-x)^2+(y-1)^2}\,dt\ =\ 0\ ,$$

implies a correctness of (3.2.6).

To prove (3.1.3) for $f(t) = M_\alpha(\mu,t)$, observe, at first, that due to (2.1.4):

$$\frac{1}{4}M_\alpha(\mu,x)^2 \leq \sup_{p\in\mathcal{P}_\alpha(\mu)}(1+p(x)^2) = \sup_{p\in\mathcal{P}_\alpha(\mu)}|p_-(x)|^2 \ \ \forall\, x \in \mathbb{R}\ ,$$

where the polynomial $p_- \in \mathcal{P}[\mathbb{C}]$ contains all zeros of the polynomial $1 + p(z)^2$ lying in $\mathbb{H}_- := \{z \in \mathbb{C} \mid \mathrm{Im} z < 0\ \}$ and $1 + p(x)^2 = |p_-(x)|^2 \ \ \forall\, x \in \mathbb{R}$. But $\|p_-\|_{L^c_\alpha(\mu)} \leq \sqrt{2}\,\|1+|p|\|_{L_\alpha(\mu)} \leq 2\sqrt{2}$ and so

$$1 \leq M_\alpha(\mu,x) \leq 2\sup_{q\in\mathcal{P}^-_\alpha(\mu)}|q(x)| \quad \forall\, x \in \mathbb{R}\ , \tag{3.2.10}$$

where $\mathcal{P}^-_\alpha(\mu) := \Big\{q \in \mathcal{P}[\mathbb{C}] \mid \Lambda_q \subset \mathbb{H}_-, \|q\|_{L^c_\alpha(\mu)} \leq 2\sqrt{2}, |q(x)| \geq 1\ \forall\, x \in \mathbb{R}\ \Big\}$. Since $\mathcal{P}^-_\alpha(\mu) \subseteq 2\sqrt{2}\mathcal{P}^c_\alpha(\mu)$, then denoting for the introduced in (3.2.9) function Φ : $\tau(t) := |\log 2\sqrt{2}\Phi(t)|$, $t \in \mathbb{R}$, we can state that for arbitrary $q \in \mathcal{P}^-_\alpha(\mu)$ function $\log|q(z)|$ is a harmonic function in $0 \leq \mathrm{Im} z \leq 1$ and in view of (3.2.8) it satisfies inequality: $\log|q(1+it)| \leq \tau(t)\ \forall\, t \in \mathbb{R}$. But (3.2.7) is true for $\psi = 2\sqrt{2}\Phi$ and that is why application of the Poisson formula to $\log|q(z)|$ gives possibility to continue inequality (3.2.10) as follows:

$$\log\frac{M_\alpha(\mu,x)}{2} \leq \sup_{q\in\mathcal{P}^-_\alpha(\mu)}\frac{1}{\pi}\int\limits_{\mathbb{R}}\frac{\log|q(1+it)|}{1+(t-x)^2}\,dt \leq \frac{1}{\pi}\int\limits_{\mathbb{R}}\frac{\tau(t)}{1+(t-x)^2}\,dt, \ \ \forall\, x \in \mathbb{R}\ . \tag{3.2.11}$$

Using Fubini theorem and equality $\int_{\mathbb{R}}(1+x^2)^{-1}[1+(t-x)^2]^{-1}\,dx = 2\pi(t^2+4)^{-1}$, we get from (3.2.11) the required inequality:

$$\int\limits_{\mathbb{R}} \frac{\log|M_\alpha(\mu,x)|}{1+x^2}\,dx \le \pi\log 2 + 4\int\limits_{\mathbb{R}} \frac{\tau(t)}{1+t^2}\,dt \ < \ \infty \ .$$

Proposition 3.1 is proved.

3.3. New version of M.G.Krein's Theorem and its corollaries.

3.3.1. Setting of a problem and M. Krein's theorem.

Let $f \in \mathcal{E}^*(\mathbb{R})$ and $\Lambda_f = \{\lambda_n\}_{n\ge1}$. By Mittag-Leffler theorem [31, v.II] there exists such sequence $\{p_n\}_{n\ge1} \subset \mathbb{Z}_0$ that

$$\sum_{n\ge1} \frac{|z|^{p_n}}{|\lambda_n|^{1+p_n}|f'(\lambda_n)|} \ < \ \infty \quad \forall\ z\in\mathbb{C} \tag{3.3.1}$$

and function

$$\frac{1}{f(z)} - \sum_{n\ge1} \frac{z^{p_n}}{\lambda_n^{p_n} f'(\lambda_n)(z-\lambda_n)}$$

is an entire function, where for $\lambda, z \in \mathbb{C}$ and positive integer p : $(z/\lambda)^p(z-\lambda)^{-1} = (z-\lambda)^{-1} + \frac{1}{\lambda} + \frac{z}{\lambda^2} + \ldots + \frac{z^{p-1}}{\lambda^p}$. Assumptiom (see (3.1.6)) $d_f < +\infty$ gives possibility to set in (3.3.1) $p_n = p \ge 0 \vee d_f \ \ \forall\ n\ge1$ and consider an entire function $\Delta_f^p(z)$ defined by (3.1.7). Just such assumption about the entire functions from more wide class (A) (see 3.1.) have been made by M. Krein in [24]. But everywhere below we will consider M. Krein's results only on the set of real entire functions all zeros of which are real.

So, for $f \in \mathcal{E}^*(\mathbb{R})$ M. Krein in [24] made an assumption $d_f < +\infty$ and considered the problem of the description of all those functions $f \in \mathcal{E}^*(\mathbb{R})$ entire function $\Delta_f^p(z)$ of which for some $p \ge 0 \vee d_f$ is a polynomial.

M. Krein in [24] proved theorem which is described detally in [29], has a self-contained proof in [23] and has been discussed also in [34, 25, 9, 11]. We will use the following its version given in [9, Th.6.1].

Proposition 3.2. *Let $f \in \mathcal{E}_s(\mathbb{R})$ and the following relations hold:*

$$\sum_{\lambda\in\Lambda_f} \frac{1}{|f'(\lambda)|} < \infty \ ; \quad \frac{1}{f(z)} = \sum_{\lambda\in\Lambda_f} \frac{1}{f'(\lambda)(z-\lambda)} \ , \ \forall\ z \in\ \mathbb{C}\setminus\Lambda_f \ .$$

Then $f \in$ Cartwright .

Such form of M. Krein's theorem requires some additional comments. It was proved in [9, Lemma 6.3] that if $f \in \mathcal{E}_s(\mathbb{R})$, $d_f < +\infty$ and for some $p \ge 0 \vee d_f$ entire function $\Delta_f^p(z)$ is a polynomial then for any polynomial $Q \in \mathcal{P}_s(\mathbb{R})$, satisfying $\Lambda_Q \cap \Lambda_f = \emptyset$, $\deg Q > p \vee \deg\Delta_f^p$, function $g := Q\cdot f$ has the property $\Delta_g^0(z) \equiv 0$. That is why under these conditions using Proposition 3.2 we get also $f \in$ Cartwright . So, taking into

account that remark from the paper [9] and also indicated in 3.1 possibility to substitute inequality (3.1.2) by (3.1.3) we can reformulate Proposition 3.2 as follows.

M. Krein Theorem.([24]) *Let $f \in \mathcal{E}_s(\mathbb{R})$ and $d_f < \infty$. If for some $p \geq 0 \vee d_f$ entire function $\Delta_f^p(z)$ is a polynomial then the following two properties hold:*

$$(3.3.2a) \qquad f \in \mathcal{E}_1(\mathbb{R}) \ ; \qquad\qquad (3.3.2b) \qquad \int\limits_{\mathbb{R}} \tfrac{|\log|f(t)||}{1+t^2}\, dt \ < \infty \quad .$$

Note the following evident properties of the quantity d_f for $f \in \mathcal{E}_s(\mathbb{R})$:

(3.3.3a) $d_{f\cdot Q} = d_f - d_Q \quad \forall\ Q \in \mathcal{P}_s(\mathbb{R}) :\ \Lambda_Q \cap \Lambda_f = \emptyset$;

(3.3.3b)$d_{\frac{f}{Q}} = d_f + d_Q \quad \forall\ Q \in \mathcal{P}_s(\mathbb{R}) :\ \Lambda_Q \subset \Lambda_f$;

In the paper [9] the following Akhiezer's [2, III.11] remark was considered: if $f \in \mathcal{E}_s(\mathbb{R})$, $d_f \leq 1$ and $\Delta_f^1(z)$ is a polynomial then $\deg\Delta_f^1 = 0$. In the Corollary 6.4 from [9] the more general result was proved: if $f \in \mathcal{E}_s(\mathbb{R})$, $d_f < \infty$ and $\Delta_f^p \in \mathcal{P}$ for some $p \geq 1 \vee d_f$, then $\deg\Delta_f^p \leq p-1$, and for $p = 0 \leq d_f$: $\Delta_f^0(z) \equiv 0$. Using this fact, Phragmen-Lindelof principle and possibility to differentiate series (3.1.7) it is easy to derive the validity of the following statement given here without proof.

Proposition 3.3. *Let $f \in \mathcal{E}_s(\mathbb{R})$, $d_f < +\infty$ and for some $p_0 \geq 0 \vee d_f$ entire function $\Delta_f^{p_0}(z)$ is a polynomial. Then (see (3.1.8))*

(3.3.4a) $\frac{1}{f_a(z)} = m_{f_a}^p(z) \quad \forall z \in \mathbb{C} \setminus \Lambda_{f_a} \quad \forall p \geq 0 \vee d_f \quad \forall a \in \mathbb{R}$;

(3.3.4b) $\frac{1}{f(z)Q(z)} = m_{f\cdot Q}^p(z) \quad \forall z \in \mathbb{C} \setminus \Lambda_{f\cdot Q} \quad \forall p \geq 0 \vee (d_f - \deg Q) \quad \forall\ Q \in \mathcal{P}_s(\mathbb{R})$: $\Lambda_Q \cap \Lambda_f = \emptyset$;

(3.3.4c) $\frac{Q(z)}{f(z)} = m_{\frac{f}{Q}}^p(z) \quad \forall z \in \mathbb{C} \setminus \Lambda_{\frac{f}{Q}} \quad \forall p \geq 0 \vee (d_f + \deg Q) \quad \forall\ Q \in \mathcal{P}_s(\mathbb{R}) :\ \Lambda_Q \subset \Lambda_f$.

where $f_a(z) := f(z+a)$, $a \in \mathbb{R}$, $z \in \mathbb{C}$.

That is why the following property of functions $f \in \mathcal{E}_s(\mathbb{R}) :\ d_f < \infty$ and $\exists p \geq 0 \vee d_f$: $\Delta_f^p \in \mathcal{P}$, is invariant with respect to the translation $f(z+a)$, $a \in \mathbb{R}$, multiplication and division on the pointed out in (3.3.4b) and (3.3.4c) polynomials.

3.3.2. Main results.

Continuing consideration of the simple properties of the functions from the class $\mathcal{E}(\mathbb{R}) \cap$ Cartwright , being started in [9, Th.6.2], we establish the following statement.

Lemma 3.1. *Let $f \in \mathcal{E}_1(\mathbb{R})$ and*

$$\sum_{\lambda \in \Lambda_f \setminus \{0\}} \frac{1}{|\lambda|} < \infty \ . \tag{3.3.5}$$

Then the following statements hold:

(3.3.5a) *if $d_f < +\infty$ and $\mathrm{co}\Lambda_f = \mathbb{R}$ then $f \in \mathcal{E}_0(\mathbb{R})$;*

(3.3.5b) *if* $f \in$ Cartwright *then* $f \in \mathcal{E}_0(\mathbb{R})$ *;*

(3.3.5c) *if* $f \in$ Cartwright *and* $\mathrm{co}\Lambda_f \neq \mathbb{R}$ *then* $\sum\limits_{\lambda\in\Lambda_f\setminus\{0\}} \frac{\log^+|\lambda|}{|\lambda|} < \infty$.

Proof of Lemma 3.1. By Hadamard's theorem $f(z) = e^{-az}P(z)$, $a \in \mathbb{R}$, $z \in \mathbb{C}$, $P(z) = \prod_{\lambda\in\Lambda_f}(1 - \frac{z}{\lambda})$ where without loss of generality we assume $0 \notin \Lambda_f$. If $a \neq 0$ then change of variables z by $-z$ allows us to consider only the case $a > 0$. It is known [42, 8.6.4., Ex.8.15] that $P \in \mathcal{E}_0$ and therefore $\exists C > 0$: $|P(z)| \leq Ce^{\frac{a}{2}x}$ $\forall\ z = x + iy$, $y \in I$, $x \geq 0$. Then $|f(x)|, |f'(x)| \leq Ce^{-\frac{a}{2}(x-1)}$ $\forall\ x \geq 1$. Such inequalities lead to a contradiction with $d_f < \infty$, if $\mathrm{co}\Lambda_f = \mathbb{R}$ and with inequality (3.1.3), if $f \in$ Cartwright . So, $f \equiv P \in \mathcal{E}_0(\mathbb{R})$ and (3.3.5a), (3.3.5b) are proved. Prove at last (3.3.5c). Change of variables $f(b \pm x)$, $b \in \mathbb{R}$, allows us to assume $\Lambda_f = \{\lambda_n\}_{n\geq 1} \subset [1, +\infty)$. Then according to (3.3.2b) for arbitrary $N \geq 1$:

$$\int\limits_{\mathbb{R}} \frac{|\log|f(x)||}{1+x^2}\,dx \ \geq \ \int\limits_{-\infty}^{0} \frac{\log P(x)}{1+x^2}\,dx \ \geq \ \sum_{n=1}^{N} \int\limits_{0}^{\infty} \frac{\log(1+\frac{x}{\lambda_k})}{1+x^2}\,dx \ \geq \ (\log 2)\cdot \sum_{n=1}^{N} \frac{\log \lambda_k}{\lambda_k} \ ,$$

what was to be proved. $\square$

Since the Lindelof's theorem [29, I] implies validity of (3.3.5) for $f \in \mathcal{E}_1(\mathbb{R})$ with $\mathrm{co}\Lambda_f \neq \mathbb{R}$, then due to Lemma 3.1 we have validity of the following implication:

$$f \in \mathcal{E}(\mathbb{R}) \cap \text{Cartwright},\ \mathrm{co}\Lambda_f \neq \mathbb{R} \ \Rightarrow \ f \in \mathcal{E}_0(\mathbb{R}) \ . \tag{3.3.6}$$

The following statement represents another version of M.G. Krein's theorem.

Theorem 3.1. *Let f be non-constant real entire function with only real and simple zeros and*

$$d_f \ := \ \inf \ \left\{ q \in \mathbb{Z} \mid \sum_{\lambda\in\Lambda_f\setminus\{0\}} \frac{1}{|f'(\lambda)|\cdot|\lambda|^{q+1}} \ < \ \infty \ \right\} < +\infty. \tag{3.3.7}$$

The following statements are equivalent:

(3.3.8a) *There exists such integer $p \geq 0 \vee d_f$ that entire function*

$$\frac{1}{f(z)} \ - \ \frac{\chi_{\Lambda_f}(0)}{f'(0)\cdot z} \ - \ \sum_{\lambda\in\Lambda_f\setminus\{0\}} \frac{z^p}{\lambda^p f'(\lambda)(z-\lambda)}$$

is a polynomial.

(3.3.8b) *Function f is an entire function of exponential type and*

$$\int\limits_{\mathbb{R}} \frac{|\ \log|f(t)|\ |}{1+t^2}\,dt \ < \infty \ .$$

(3.3.8c) *If $\mathrm{co}\Lambda_f = \mathbb{R}$, then f is an entire function of exponential type, but if $\mathrm{co}\Lambda_f \neq \mathbb{R}$, then f is an entire function of minimal exponential type.*

Implication $(3.3.8a)\ \Rightarrow\ (3.3.8b)$ coincides identically with M. Krein's theorem. Implication $(3.3.8b)\ \Rightarrow\ (3.3.8a)$ has been proved by L.de Branges in his famous paper [12, Lemma 2], where one need to take $G \in \mathcal{P}$ and observe that $|F(iy)|$ tends to infinity faster than any exponential function. Implication $(3.3.8b)\ \Rightarrow\ (3.3.8c)$ follows from (3.3.6). Implication $(3.3.8c)\ \Rightarrow\ (3.3.8a)$ for the entire functions of minimal exponential type was proved in the master's thesis of Henrik L. Pederson at University of Copenhagen and can be found in [9] as Theorem 6.6. In view of Lemma 3.1 it is remained to prove only those part of $(3.3.8c)\ \Rightarrow\ (3.3.8a)$ where $f \in \mathcal{E}_1(\mathbb{R}) \setminus \mathcal{E}_0(\mathbb{R})$, $\mathrm{co}\Lambda_f = \mathbb{R}$ and so by (3.3.5a) $\sum_{\lambda\in\Lambda_f\setminus\{0\}} \frac{1}{|\lambda|} = \infty$. This part can be easily derived from the following theorem which will be proved in 3.5.

Theorem 3.2. *Let real entire transcendental function f has only real zeros and taking into account their multiplicity $\{\lambda_k \mid Q < k < P\} := \Lambda_f\setminus\{0\}$, $P, Q \in \mathbb{Z}\cup\{\pm\infty\}$, $\lambda_k \le \lambda_{k+1}\ \ \forall\ Q < k < P-1$. Let also exist such increasing sequences of positive real numbers R_n, r_n , $n \ge 1$, that $R_n, r_n \to +\infty$, $n \to \infty$, and*

$$f(z) = z^m \frac{f^{(m)}(0)}{m!} \cdot \lim_{n\to\infty} \prod_{\lambda\in(\Lambda_f\setminus\{0\})\cap(-r_n,R_n)} \left(1 - \frac{z}{\lambda}\right) \quad \forall\ z \in \mathbb{C}\ , \tag{3.3.9}$$

where $m \in \mathbb{Z}_0$.

Then $f \in \mathcal{LP}^0_{II}$ and there exist such sequences of integers $p_N, q_N : Q < q_N < p_N < P$, $N \ge 1$, that the polynomial divisors of the function f which have the following form:

$$P_N(z) := \frac{f^{(m)}(0)}{m!} \cdot z^m \cdot \prod_{k=q_N}^{p_N} \left(1 - \frac{z}{\lambda_k}\right) \tag{3.3.10}$$

converge to $f(z)$ uniformly on any compact subset of $\mathbb{C}$ and satisfy conditions:

(3.3.11a) $(-N, N) \cap \Lambda_f \subseteq \{\lambda_k\}_{k=q_N}^{p_N}$;

(3.3.11b) $|P_N(x)| \ge \frac{1}{e} \cdot |f(x)| \quad \forall\ x \in [\lambda_{q_N}, \lambda_{p_N}]$;

(3.3.11c) $|P_N^{(m_k)}(\lambda_k)| \ge \frac{1}{e} \cdot |f^{(m_k)}(\lambda_k)|\ \ \forall\ q_N \le k \le p_N$,

where $m_k \ge 1$ denotes the multiplicity of zero $\lambda_k \in \Lambda_f \setminus \{0\}\ \ \forall\ Q < k < P$ (in terms of the set Λ_f this means that m_k is a number of the equal to λ_k elements in Λ_f).

Remark 3.1.(*Sense of the condition (3.3.8c)*) Representation (3.3.9) means in particular that function f can be obtained not only as a limit of some sequence of real polynomials with real zeros but as a limit of its polynomial divisors. Consider an arbitrary $f \in \mathcal{E}_s(\mathbb{R}) \cap \mathcal{E}_1$ with $d_f < +\infty$ and clarify in what cases that function cannot be represented as a limit of its polynomial divisors. If $f \in \mathcal{E}_0$ then (3.3.9) is a corollary of Lindelof's theorem. Let $f \in \mathcal{E}_1 \setminus \mathcal{E}_0$.

If $\sum\limits_{\lambda\in\Lambda_f\setminus\{0\}} \frac{1}{|\lambda|} = \infty$ then once more by Lindelof's theorem $\mathrm{co}\Lambda_f = \mathbb{R}$, $\sum\limits_{\lambda\in\Lambda_f,\lambda>0} \frac{1}{|\lambda|} = \sum\limits_{\lambda\in\Lambda_f,\lambda<0} \frac{1}{|\lambda|} = +\infty$ and using the Hadamard's theorem we get for some $a \in \mathbb{R}$, $m \in \mathbb{Z}_0$ and any $R, r > 0$:

$$\frac{m!f(z)}{f^{(m)}(0)z^m} = e^{-az} \cdot \prod_{\lambda \in \Lambda_f \setminus \{0\}} \left(1 - \frac{z}{\lambda}\right) e^{\frac{z}{\lambda}} = e^{(\delta_f(r,R)-a)z} \cdot f_{r,R}(z) \cdot \prod_{\lambda \in (\Lambda_f \setminus \{0\}) \cap (-r,R)} \left(1 - \frac{z}{\lambda}\right) ,$$

where $f_{r,R}(z) := \prod_{\lambda \in \Lambda_f \setminus (-r,R)} \left(1 - \frac{z}{\lambda}\right) e^{\frac{z}{\lambda}}$, $\delta_f(r,R) := \sum_{\lambda \in (\Lambda_f \setminus \{0\}) \cap (-r,R)} \frac{1}{\lambda}$. Choosing two sequences r_n, R_n , $n \geq 1$, so that $r_n, R_n \to +\infty$, $\delta_f(r_n, R_n) \to a$, $n \to \infty$, we get representation (3.3.9).

If now $\sum_{\lambda \in \Lambda_f \setminus \{0\}} \frac{1}{|\lambda|} < \infty$ then by (3.3.5a) $\mathrm{co}\Lambda_f \neq \mathbb{R}$ and by virtue of Hadamard's theorem $f(z) = e^{az} f_0(z)$, $f_0 \in \mathcal{E}_0$, $a \in \mathbb{R} \setminus \{0\}$. Our condition $d_f < +\infty$ for functions of such kind indicates only that $a > 0$, if $\sup \Lambda_f = +\infty$, and $a < 0$, if $\inf \Lambda_f = -\infty$. But in both cases f of this kind cannot be represented as a limit of some its polynomial divisors. Just that class of entire functions has been excluded by condition (3.3.8c).

That is why any $f \in \mathcal{E}_s(\mathbb{R})$ which satisfies (3.3.8c) and $d_f < +\infty$ can be represented in the form of (3.3.9). □

Proof of Theorem 3.1. As it was noted above it is remained to prove only implication (3.3.8c) $\Rightarrow$ (3.3.8a) where in view of Remark 3.1 one can apply to the considered function f the Theorem 3.2. But firstly we multiply f on the polynomial Q satisfying (3.3.3a) in order to obtain $d_g \leq -1$ for $g := f \cdot Q$. Approximating g by polynomials P_N from Theorem 3.2 we will get by (3.3.11c) for arbitrary $z \in \mathbb{C} \setminus \Lambda_g$ and $R > 0$:

$$\left| \frac{1}{P_N(z)} - \sum_{\lambda \in \Lambda_{P_N} \cap (-R,R)} \frac{1}{P_N'(\lambda)(z-\lambda)} \right| = \left| \sum_{\lambda \in \Lambda_{P_N} \setminus (-R,R)} \frac{1}{P_N'(\lambda)(z-\lambda)} \right| \leq$$

$$\leq e \left(\sum_{\lambda \in \Lambda_g \setminus (-R,R)} \frac{1}{|g'(\lambda)|} \right) \cdot \sup_{\lambda \in \Lambda_g \setminus (-R,R)} \frac{1}{|z-\lambda|} \tag{3.3.12}$$

and passing to the limit as $N \to \infty$, we obtain for every $z \in \mathbb{C} \setminus \Lambda_g$:

$$\left| \frac{1}{g(z)} - \sum_{\lambda \in \Lambda_g \cap (-R,R)} \frac{1}{g'(\lambda)(z-\lambda)} \right| \to 0, \ R \to +\infty , \tag{3.3.13}$$

from where with the help of Proposition 3.3 we derive the required property (3.3.8a) for the function f . □

Theorems 3.1, 3.2 and Remark 3.1 give possibility to characterize Hamburger and Krein classes of entire functions (see 3.1) in terms of the behavior of the *derivative numbers* $\{f'(\lambda)\}_{\lambda \in \Lambda_f}$ of the entire function f .

Corollary 3.1.

1. *Entire function $f(z)$ belongs to the Krein class $\mathcal{K}$ if and only if it has the following properties:*

(3.3.14a) f *is a real function with only real and simple zeros* Λ_f ;

(3.3.14b) *if* $\mathrm{co}\Lambda_f = \mathbb{R}$ *then* f *is of exponential type, but if* $\mathrm{co}\Lambda_f \neq \mathbb{R}$ *then* f *is of minimal exponential type* ;

(3.3.14c) $\sum\limits_{\lambda\in\Lambda_f} \frac{1}{(1+\lambda^2)|f'(\lambda)|} < \infty$.

2. *Entire function* $f(z)$ *belongs to the Hamburger class* $\mathcal{H}$ *if and only if it has the following properties:*

(3.3.15a) f *is a real function with only real and simple zeros* Λ_f ;

(3.3.15b) *if* $\mathrm{co}\Lambda_f = \mathbb{R}$ *then* f *is of exponential type, but if* $\mathrm{co}\Lambda_f \neq \mathbb{R}$ *then* f *is of minimal exponential type* ;

(3.3.15c) $\lim\limits_{\substack{|\lambda|\to\infty \\ \lambda\in\Lambda_f}} \frac{|\lambda|^n}{|f'(\lambda)|} = 0 \quad \forall\, n \in \mathbb{Z}_0$. $\triangle$

It should be noted that entire functions satisfying conditions (3.3.14a) and (3.3.14b) form a sufficiently large subset of the second Laguerre-Pólya class $\mathcal{LP}_{II}$ of entire functions and each of them can be represented as follows:

$$f(z) = \begin{cases} c\cdot z^q \cdot \prod\limits_{k\geq 1}\left(1-\frac{z}{\lambda_k}\right) \ ; & \sum\limits_{k\geq 1}\frac{1}{|\lambda_k|} < \infty \ , \qquad \text{if } \mathrm{co}\,\{\lambda_k\}_{k\geq 1} \neq \mathbb{R} \ ; \\ c\cdot z^q\cdot e^{az}\cdot \prod\limits_{k\geq 1}\left(1-\frac{z}{\lambda_k}\right)e^{\frac{z}{\lambda_k}} \ ; & a\in\mathbb{R},\ \sum\limits_{k\geq 1}\frac{1}{\lambda_k^2} < \infty \ , \quad \text{if } \mathrm{co}\,\{\lambda_k\}_{k\geq 1} = \mathbb{R} \ ; \end{cases} \tag{3.3.16}$$

where $c, \lambda_k \in \mathbb{R}\setminus\{0\} \ \ \forall\ k\geq 1$, $\lambda_k \neq \lambda_m$, if $k\neq m$, $k,m\geq 1$, $q\in\mathbb{Z}_0$. But if we impose on the function f from that class only one condition on their derivative numbers: $d_f < +\infty$, or, what is the same,

$$\varlimsup_{\substack{|\lambda|\to\infty \\ \lambda\in\Lambda_f}} \frac{\log^+\frac{1}{|f'(\lambda)|}}{\log|\lambda|} < +\infty \ , \tag{3.3.17}$$

then by Theorem 3.1 we can conclude that this function f will be an element of Cartwright class, what in the case $\mathrm{co}\,\{\lambda_k\}_{k\geq 1} \neq \mathbb{R}$ means by virtue of Lemma 3.1 that $\sum_{\lambda\in\Lambda_f\setminus\{0\}} \frac{\log^+|\lambda_k|}{|\lambda_k|} < \infty$ and in the case $\mathrm{co}\,\{\lambda_k\}_{k\geq 1} = \mathbb{R}$ gives an existence of the limit $\delta_f := \lim\limits_{R\to+\infty} \sum_{\lambda\in(-R,R)\cap\left(\Lambda_f\setminus\{0\}\right)} \frac{1}{\lambda}$, equality $a+\delta_f = 0$ and also an existence and equality of two finite limits: $\lim\limits_{R\to+\infty} \frac{\mathrm{card}\left(\Lambda_f\cap[0,R]\right)}{R} = \lim\limits_{R\to+\infty} \frac{\mathrm{card}\left(\Lambda_f\cap[-R,0]\right)}{R}$ (see [29, V.4, Th.11]).

Besides that Theorem 3.2 and Remark 3.1 allow us to establish direct and inverse polynomial approximation theorem for the entire functions from Hamburger and Krein classes. It should be noted here that since both classes $\mathcal{H}$ and $\mathcal{K}$ are subsets of the second Laguerre-Pólya class $\mathcal{LP}_{II}$ then every function from these classes can be approximated [18, III, Th.3.2] by real polynomials with real zeros only with respect to the topology $\tau_{\mathcal{E}}$.

COROLLARY 3.2.

1. *Entire function $f(z)$ belongs to the Krein class $\mathcal{K}$ if and only if there exists the sequence of real polynomials $\{P_n\}_{n\geq 1}$ with only real and simple zeros $\{\Lambda_{P_n}\}_{n\geq 1}$ which uniformly on any compact subset of the complex plane converges to the function f and for some does not depending on n constant $C > 0$ the following inequality holds:*

$$\sum_{\lambda\in\Lambda_{P_n}} \frac{1}{(1+\lambda^2)|P_n'(\lambda)|} \leq C \ \forall\, n \geq 1 \, . \tag{3.3.18}$$

2. *Entire function $f(z)$ belongs to the Hamburger class $\mathcal{H}$ if and only if there exists the sequence of real polynomials $\{P_n\}_{n\geq 1}$ with only real and simple zeros $\{\Lambda_{P_n}\}_{n\geq 1}$ which uniformly on any compact subset of the complex plane converges to the function f and for some does not depending on n function $w : \mathbb{R} \to (0,+\infty)$, $\sup\limits_{x\in\mathbb{R}} |x|^n w(x) < +\infty$ $\forall\, n \in \mathbb{Z}_0$, the following inequality holds:*

$$|P_n'(\lambda)| \geq \frac{1}{w(\lambda)} \quad \forall\, \lambda \in \Lambda_{P_n} \quad \forall\, n \geq 1 \, . \tag{3.3.19}$$

Moreover, in both items the approximating polynomial sequence can be chosen as the subset of all polynomial divisors of the function $f(z)$.

PROOF OF COROLLARY 3.2. Necessity follows easily from the Theorems 3.1, 3.2 and Remark 3.1.

Sufficiency. 1. Multiplying each polynomial P_n , $n \geq 1$, on the polynomial of the second degree $Q_2 \in \mathcal{P}_s(\mathbb{R})$ satisfying $\Lambda_{Q_2} \cap \Lambda_f = \emptyset$ we by Hurwitz's theorem obtain $\Lambda_{Q_2} \cap \Lambda_{P_n} = \emptyset$ for sufficiently large n . Resulting polynomial sequence converges to $g := Q_2 \cdot f$, where $d_g \leq -1$. Performing similar to (3.3.12) estimate with $R > 2|z|$ we get after passage to the limit as $n \to \infty$ relation (3.3.13) which by Proposition 3.3 yields $f \in \mathcal{K}$.

2. Here one need to use Pólya-Laguerre [29, VIII.1, Th.3] theorem according to which for some does not depending on n constant $M > 0$: $\sum\limits_{\lambda\in\Lambda_{P_n}} \frac{1}{1+\lambda^2} \leq M \ \ \forall\, n \geq 1$. We can perform a similar to (3.3.12) estimate from where taking into account the following corollary of (3.3.19):

$$\sum_{\lambda\in\Lambda_{P_n}\setminus(-R,R)} \frac{1}{|P_n{}'(\lambda)|} \leq \sum_{\lambda\in\Lambda_{P_n}\setminus(-R,R)} \frac{(1+\lambda^2)w(\lambda)}{1+\lambda^2} \leq M \left\|(1+x^2)w\right\|_{C(\mathbb{R})} \ ,$$

one can easily obtain for $R > 2|z|$ (3.3.13) and then by (3.3.19) $f \in \mathcal{H}$. □

3.4. Strictly normal polynomial families.

3.4.1. MAIN RESULTS. Recall (see 3.1) that $\mathcal{P}^*(\mathbb{R})$ denotes the set of real polynomials P with only real and simple zeros and $P(0) = 1$.

Let $G \subseteq \mathcal{P}^*(\mathbb{R})$ be a normal family of polynomials (see 3.1). Making use a proof by contradiction it is easy to derive succeeding the proof of Pólya-Laguerre [18, III,

Th.3.3] theorem that $\lambda_1(G) := \sup_{P\in G} |P'(0)| = \sup_{P\in G} \left|\sum_{\lambda\in\Lambda_P} \frac{1}{\lambda}\right| < \infty$ and $\lambda_2(G) := \sup_{P\in G} (P'(0)^2 - P''(0)) = \sup_{P\in G} \sum_{\lambda\in\Lambda_P} \frac{1}{\lambda^2} < \infty$. Conversely, if these quantities are finite then an obvious inequality $|(1-z)e^z| \le e^{\frac{1}{2}|z|^2} \ \forall \ z \in \mathbb{C}$ implies $|P(z)| \le \exp\left(M_1|z| + \frac{1}{2}M_2|z|^2\right) \ \forall \ z \in \mathbb{C}$, i.e. by Vitali's classical compactness theorem G is a normal set. Thus, we have proved the following statement (see also [29, VIII.1]).

PROPOSITION 3.4. *Polynomial family $G \subset \mathcal{P}^*(\mathbb{R})$ is a normal set if and only if the following two conditions hold:*

$$(3.4.1a) \ \ \lambda_1(G) := \sup_{P\in G} \left|\sum_{\lambda\in\Lambda_P} \frac{1}{\lambda}\right| < \infty \ ; \quad (3.4.1b) \ \ \lambda_2(G) := \sup_{P\in G} \sum_{\lambda\in\Lambda_P} \frac{1}{\lambda^2} < \infty \ .$$

By well-known Pólya-Laguerre theorem [18, III, Th.3.3] the closure of normal set $G \subset \mathcal{P}^*(\mathbb{R})$ denoted as $\mathrm{Close}_{\mathcal{E}}G$ is a compact subset of the second Laguerre-Pólya class of the entire functions $\mathcal{LP}_{II}$ [18, III, Def.3.1].

DEFINITION 3.1. *Normal family $G \subset \mathcal{P}^*(\mathbb{R})$ is said to be a strictly normal polynomial family if for any convergent with respect to the topology $\tau_{\mathcal{E}}$ sequence $\{P_n\}_{n\ge1} \subseteq G$ satisfying $\lim_{n\to\infty} \deg P_n = \infty$ an entire function $\lim_{n\to\infty} P_n(z)$ is transcendental.*

In terms of the introduced by P. Painleve [26, II.29] notion of upper limit of the sequence $\{A_n\}_{n\ge1}$ of subsets of some topological space with topology τ :

$$\mathrm{Ls}_{n\to\infty}A_n := \bigcap_{n\ge1} \mathrm{Close}_{\tau}\left(\bigcup_{k\ge n} A_k\right) ,$$

where $\mathrm{Close}_{\tau}A$ denotes the closure of A in the considered topological space, Definition 3.1 means that $G \subset \mathcal{P}^*(\mathbb{R})$ is a strictly normal polynomial family if and only if $G \cap \mathrm{Ls}_{n\to\infty}(G_{n+1} \setminus G_n) = \emptyset$, or, what is the same,

$$G \cap \left[\bigcap_{n\ge1} \mathrm{Close}_{\mathcal{E}}(G \setminus G_n)\right] = \emptyset \ , \quad G_n := \{P \in G \ | \ \deg P \le n\} \ , \quad n \ge 1 \ . \qquad (3.4.2)$$

It is easy to verify that the closure $\mathrm{Close}_{\mathcal{E}}G$ of the strictly normal polynomial family G aside from the compactness property have one more characteristic one: the set G being considered as a subset of the topological space $\mathrm{Close}_{\mathcal{E}}G$ with induced topology (from the whole space of all entire functions with topology $\tau_{\mathcal{E}}$) is an open set, i.e.

$$\mathrm{Close}_{\mathcal{E}}\left((\mathrm{Close}_{\mathcal{E}}G) \setminus G\right) = (\mathrm{Close}_{\mathcal{E}}G) \setminus G \ . \qquad (3.4.3)$$

In other words (3.4.3) means that any convergent sequence of the transcendental entire functions $\{f_n\}_{n\ge1} \subseteq \mathrm{Close}_{\mathcal{E}}G$ can have in capacity of its limit only also transcendental entire function or, what is the same, the set $(\mathrm{Close}_{\mathcal{E}}G) \setminus G$ of all transcendental entire functions from $\mathrm{Close}_{\mathcal{E}}G$ is a closed and hence, compact set. That is why the closure of any strictly normal polynomial set G with respect to topology $\tau_{\mathcal{E}}$ generates at once two compact sets: $\mathrm{Close}_{\mathcal{E}}G$ and $(\mathrm{Close}_{\mathcal{E}}G) \setminus G$.

In contrast to normality criterion of the Proposition 3.4 we will be interested here in those sufficient conditions for the normality and strictly normality of the polynomial set $G \subset \mathcal{P}^*(\mathbb{R})$ which can be formulated in terms of derivative numbers $\{P'(\lambda)\}_{\lambda\in\Lambda_P}$ of the polynomials from that set and which would give possibility to exclude condition of the type (3.4.1a) at all and to make condition of the type (3.4.1b) a little weaker.

Lemma 3.2. *For arbitrary finite constants $\alpha, \beta, \gamma, \delta_\alpha, \delta_\beta > 0$ the set*

$$\left\{ P \in \mathcal{P}^*(\mathbb{R}) \ \mid \ \sum_{\lambda\in\Lambda_P} e^{-\alpha|\lambda|} \le \delta_\alpha \ ; \ \ |P'(\lambda)| \ge \delta_\beta e^{-\beta|\lambda|}|\lambda|^{-1-\gamma} \ \forall \ \lambda \in \Lambda_P \right\} \tag{3.4.4}$$

is normal with respect to topology of the uniform convergence on all compact subsets of the complex plane (see 3.1).

The sequence of polynomials $\{1 - nx\}_{n\ge1}$ shows that the set (1) for $\gamma = 0$ is not normal. Denote by $C^*_+(\mathbb{R})$ the family of all positive functions from $C^*(\mathbb{R})$ (see 2.1), i.e. the set of all upper semicontinuous functions $\mu : \mathbb{R} \to (0, +\infty)$, satisfying conditions: $\|x^n \cdot \mu\|_{C(\mathbb{R})} < \infty \ \ \forall \ n \in \mathbb{Z}_0$.

Theorem 3.3. *For any $\mu \in C^*_+(\mathbb{R})$ and arbitrary finite constants $\alpha, \gamma, \delta_\alpha > 0$ the set*

$$\left\{ P \in \mathcal{P}^*(\mathbb{R}) \ \mid \ \sum_{\lambda\in\Lambda_P} e^{-\alpha|\lambda|} \le \delta_\alpha \ ; \ \ |P'(\lambda)| \ge \frac{1}{\mu(\lambda)|\lambda|^{1+\gamma}} \ \forall \ \lambda \in \Lambda_P \right\} \tag{3.4.5}$$

is a strictly normal polynomial set (see Definition 3.1).

Let $\mathcal{H}^* := \{f \in \mathcal{H} \ \mid f(0) = 1\}$ ($\mathcal{P}^*(\mathbb{R}) \subset \mathcal{H}^*$) and for $\mu \in C^*_+(\mathbb{R})$, $\gamma, C_\gamma \in (0, +\infty)$, denote

$$\mathcal{P}^*_{\mathcal{H}}(C_\gamma, \mu) := \left\{ P \in \mathcal{P}^*(\mathbb{R}) \ \mid \ \sum_{\lambda\in\Lambda_P} e^{-|\lambda|} \le C_\gamma \ ; \ \ |P'(\lambda)| \ge \frac{1}{\mu(\lambda)|\lambda|^{1+\gamma}} \ \forall \ \lambda \in \Lambda_P \right\} . \tag{3.4.6}$$

The set (3.4.6) by Theorem 3.3 is a strictly normal polynomial set. In addition, by virtue of Corollary 3.2: $\mathcal{H}^*(C_\gamma, \mu) := \mathrm{Close}_{\mathcal{E}}\mathcal{P}^*_{\mathcal{H}}(C_\gamma, \mu) \subseteq \mathcal{H}^*$ and so:

$$\mathcal{H}^*(C_\gamma, \mu) = \left\{ f \in \mathcal{H}^* \ \mid \ \sum_{\lambda\in\Lambda_f} e^{-|\lambda|} \le C_\gamma \ ; \ \ |f'(\lambda)| \ge \frac{1}{\mu(\lambda)|\lambda|^{1+\gamma}} \ \forall \ \lambda \in \Lambda_f \right\} . \tag{3.4.7}$$

The set (3.4.7) being a compact subset of the Hamburger class of entire functions $\mathcal{H}$ possesses due to (3.4.3) the following property.

Corollary 3.3. *Let $\mathcal{H}_\infty$ denote the set of transcendental entire functions from the Hamburger class $\mathcal{H}$ and for $\mu \in C^*_+(\mathbb{R})$, $\gamma, C_\gamma \in (0, +\infty)$:*

$$\mathcal{H}^*_\infty(C_\gamma, \mu) := \mathcal{H}_\infty \cap \mathcal{H}^*(C_\gamma, \mu) . \tag{3.4.8}$$

Then the set $\mathcal{H}^*_\infty(C_\gamma,\mu)$ *is a compact subset of* $\mathcal{H}_\infty$ *and*

$$\mathcal{H}^*_\infty(C_\gamma,\mu)=\left\{ f\in\mathcal{H}_\infty \mid f(0)=1, \sum_{\lambda\in\Lambda_f} e^{-|\lambda|}\le C_\gamma;\ |f'(\lambda)|\ge \frac{1}{\mu(\lambda)|\lambda|^{1+\gamma}}\ \forall\lambda\in\Lambda_f \right\}. \tag{3.4.9}$$

It should be noted at last that the statements of Lemma 3.2, Theorem 3.3 and Corollary 3.3 remain valid if we substitute the conditions on the derivative numbers of the entire functions f in (3.4.4, 5, 6, 7, 9) by the following ones:

$$\sum_{\lambda\in\Lambda_f}\frac{1}{\mu(\lambda)|\lambda|^\beta|f'(\lambda)|^\gamma}\ \le\ C(\beta,\gamma)\ ,\qquad \beta>\gamma>0\ . \tag{3.4.10}$$

Such substitution is possible because for any $\mu\in C^*_+(\mathbb{R})$: $(\mu)^\alpha\in C^*_+(\mathbb{R})$ $\forall\ \alpha\ >0$, and an arbitrary subset of normal or strictly normal polynomial set possesses also the corresponding property.

3.4.2. Proof of Lemma 3.2. Let $M\ge\alpha+\beta+1$ and P be an arbitrary polynomial of the defined by (3.4.4) polynomial family. Since for $z\in\mathbb{C}\setminus\{\alpha_n\}_{n\ge1}$, $\alpha_n:=\frac{\pi}{M}(n-\frac12)$, $n\in\mathbb{Z}$, the following equality holds:

$$\frac{1}{\cosh(Mz)}=1+\sum_{n\ge1}\frac{(-1)^n}{M\cdot\alpha_n}\left(\frac{z}{z+i\alpha_n}+\frac{z}{z-i\alpha_n}\right)\ ,$$

then, denoting $\{\lambda_k\}_{k=1}^n:=\Lambda_P$, $0<|\lambda_1|\le|\lambda_2|\le\ldots\le|\lambda_m|\le1\le|\lambda_{m+1}|\le\ldots\le|\lambda_N|$, $0\le m\le N$, ($m=0$, if $|\lambda_1|>1$) we obtain for $z\in\mathbb{C}\setminus\left(\Lambda_P\cup\{\alpha_n\}_{n\ge1}\right)$:

$$\Phi(z):=\frac{1}{P(z)\cdot\cosh(Mz)}=\sum_{k=1}^N\frac{1}{P'(\lambda_k)\cosh(M\lambda_k)}\frac{1}{z-\lambda_k}+\frac{i}{M}\sum_{n\ge1}(-1)^n\times$$

$$\times\left[\frac{1}{P(i\alpha_n)(z-i\alpha_n)}+\frac{1}{P(-i\alpha_n)(z+i\alpha_n)}\right]\ . \tag{3.4.11}$$

Differentiating equality (3.4.11) we have $\Phi'(0)=-P'(0)$, $\Phi''(0)=2P'(0)^2-P''(0)-M^2$ and

$$P'(0)\ =\ \sum_{k=1}^N\frac{1}{\lambda_k^2\cdot P'(\lambda_k)\cosh(M\lambda_k)}+\frac{i}{M}\sum_{n\ge1}\frac{(-1)^n}{\alpha_n^2}\left(\frac{1}{P(-i\alpha_n)}-\frac{1}{P(i\alpha_n)}\right)\ ,$$

$$P''(0)+M^2-2P'(0)^2=\sum_{k=1}^N\frac{1}{\lambda_k^3\cdot P'(\lambda_k)\cosh(M\lambda_k)}+\frac{1}{M}\sum_{n\ge1}\frac{(-1)^n}{\alpha_n^3}\left[\frac{1}{P(i\alpha_n)}+\frac{1}{P(-i\alpha_n)}\right],$$

from where taking into account $|P(i\lambda)|\ge1\ \ \forall\ \lambda\ \in\mathbb{R}$ and $\sum_{n\ge1}\frac{1}{(n-0{,}5)^3}<\frac{\pi^3}{2}$, we derive

$$|P'(0)|\ \le\ M+\sum_{k=1}^N\frac{1}{\lambda_k^2\cdot|P'(\lambda_k)|\cosh(M\lambda_k)}\ , \tag{3.4.12a}$$

$$2P'(0)^2-P''(0)\ \le\ 2M^2+\sum_{k=1}^N\frac{1}{|\lambda_k|^3\cdot|P'(\lambda_k)|\cosh(M\lambda_k)}\ . \tag{3.4.12b}$$

If $m=0$, then equalities (3.4.12a), (3.4.12b) and conditions (3.4.4) give estimates of $|P'(0)| = \left|\sum\limits_{k=1}^{N}\frac{1}{\lambda_k}\right|$ and $P'(0)^2 - P''(0) = \sum\limits_{k=1}^{N}\frac{1}{\lambda_k^2}$ depending on five constants of Lemma 3.2 only.

If $m\ge 1$ then (3.4.12a) and (3.4.4) yield:

$$\left|\sum_{k=1}^{N}\frac{1}{\lambda_k}\right| \le M + \frac{2}{\delta_\beta}\sum_{k=1}^{m}\frac{1}{|\lambda_k|}|\lambda_k|^{\gamma}e^{-(1+\alpha)|\lambda_k|} + \frac{2}{\delta_\beta}\sum_{k=m+1}^{N}|\lambda_k|^{\gamma-1}e^{-(1+\alpha)|\lambda_k|} \le$$

$$\le M + 2\frac{\delta_\alpha}{\delta_\beta}\frac{1}{|\lambda_1|} + 2\frac{\delta_\alpha}{\delta_\beta}\left(\frac{\gamma}{e}\right)^{\gamma}$$

and therefore, using inequality $\log(1+x)\le x\ \ \forall\ x>-1$, we get:

$$\frac{\delta_\beta e^{-\beta}}{|\lambda_1|^{1+\gamma}} \le |P'(\lambda_1)| = \frac{1}{|\lambda_1|}\prod_{k=2}^{N}\left(1-\frac{\lambda_1}{\lambda_k}\right) \le \frac{e}{|\lambda_1|}e^{-\lambda_1\left(\sum\limits_{k=1}^{N}\frac{1}{\lambda_k}\right)} \le$$

$$\le \frac{e}{|\lambda_1|}\exp\left[M + 2\frac{\delta_\alpha}{\delta_\beta}\left(1+\left(\frac{\gamma}{e}\right)^{\gamma}\right)\right] .$$

That is why there exists such constant $\delta>0$ depending on constants (3.4.4) only that $|\lambda_1|\ge\delta$. Since among all zeros of P zero λ_1 has a minimal absolute value then it follows from (3.4.12a), (3.4.12b) and conditions (3.4.4) that there exist the estimates of $\left|\sum\limits_{k=1}^{N}\frac{1}{\lambda_k}\right|$ and $\sum\limits_{k=1}^{N}\frac{1}{\lambda_k^2}$ depending on $\alpha,\beta,\gamma,\delta_\alpha,\delta_\beta>0$ only. This means that conditions of the Proposition 3.4 are fulfilled and so the set (3.4.4) is normal.

3.4.3. Proof of Theorem 3.3. Denote defined in (3.4.5) set by G . Since μ is uniformly bounded on the whole real axis then by Lemma 3.2 G is normal and due to Proposition 3.4 : $M := \lambda_1(G)\vee\lambda_2(G) < +\infty$.

Consider an arbitrary convergent to some entire function f polynomial sequence $\{P_n\}_{n\ge1}\subseteq G$, with $\lim_{n\to\infty}\deg P_n = \infty$. Denote $\left\{\lambda_k^{(n)}\right\}_{k=1}^{r_n} := \Lambda_{P_n}$, $\frac{1}{M}\le|\lambda_1^{(n)}|\le |\lambda_2^{(n)}|\le\ldots\le|\lambda_{r_n}^{(n)}|<\infty\ \ \forall\ n\ge1$. By Hurwitz's theorem for arbitrary $p\ge1$ the sequence $\left\{\lambda_p^{(n)}\right\}_{n\ge n_p}$, $r_{n_p}\ge p$, has finite or infinite limit. Let for $n\ge n_p$:

$$P_{n,p}(x) := \frac{P_n(x)}{\Delta_{n,p}(x)}\ ,\quad \Delta_{n,p}(x) := \left(1-\frac{x}{\lambda_1^{(n)}}\right)\left(1-\frac{x}{\lambda_2^{(n)}}\right)\cdot\ldots\cdot\left(1-\frac{x}{\lambda_{p-1}^{(n)}}\right)\ ,$$

where $p\ge2,\ \Delta_{1,n}(x)\equiv1$. Then for all $p\le k\le r_n$, $n\ge n_p$:

$$|P'(\lambda_k^{(n)})| = \left|\Delta_{p,n}(\lambda_k^{(n)})\right|\cdot\left|P'_{n,p}(\lambda_k^{(n)})\right| \le \left(1+M|\lambda_k^{(n)}|\right)^{p-1}\cdot\left|P'_{n,p}(\lambda_k^{(n)})\right| .$$

Using (3.4.5) and decomposition formula of $P_{n,p}(z)^{-1}$ on the simple fractions we get:

$$1\le\sum_{k=p}^{r_n}\frac{1}{|P'_{n,p}(\lambda_k^{(n)})|}\frac{1}{|\lambda_k^{(n)}|} \le \sum_{k=p}^{r_n}\mu(\lambda_k^{(n)})|\lambda_k^{(n)}|^{\gamma} \le \frac{M}{|\lambda_p^{(n)}|}\left\|x^{1+\gamma}(1+x^2)^p\mu\right\|_{C(\mathbb{R})} .$$

That is why for arbitrary $p \geq 1$ the sequence $\left\{\lambda_p^{(n)}\right\}_{n \geq n_p}$ has a finite limit $\lambda_p \in \Lambda_f$ and since by (3.4.5) function f has only simple zeros then all numbers $\{\lambda_p\}_{p\geq 1}$ are distinct and hence, f is a transcendental entire function.

3.5. Proof of Theorem 3.2. It is easy to verify that existence of the indicated by theorem polynomials is invariant with respect to the change of variables of the kind $f(a \pm x)$, $a \in \mathbb{R}$, and inequality (3.3.11c) can be obtained from (3.3.11b) by division it on $\left(1 - \frac{x}{\lambda_k}\right)^{m_k}$. Thus, to prove the theorem it is sufficient to show the validity of (3.3.11a), (3.3.11b) under conditions $0 \notin \Lambda_f$, $f(0) = 1$ and if $\mathrm{co}\Lambda_f \neq \mathbb{R}$ then without loss of generality $\Lambda_f \subset (0, +\infty)$. That is why in the case $\mathrm{co}\Lambda_f \neq \mathbb{R}$ (3.3.9) implies $\sum\limits_{\lambda\in\Lambda_f} \frac{1}{\lambda} < \infty$, $f \in \mathcal{E}_0(\mathbb{R})$ and for $N > \min \Lambda_f$ polynomials $P_N(x) := \prod\limits_{\lambda\in\Lambda_f\cap(0,N)} \left(1 - \frac{x}{\lambda}\right)$ satisfy all conditions of the theorem.

Let now $\mathrm{co}\Lambda_f = \mathbb{R}$. Then (3.3.9) for $z = i$ means $\sum\limits_{\lambda\in\Lambda_f} \frac{1}{\lambda^2} < \infty$ and then it is easy to derive from the (3.3.9) at some $z \in \mathbb{R}\setminus\{0\}$, $f(\pm z) \neq 0$, an existence of finite limit $a := \lim\limits_{p\to\infty} \sum\limits_{\lambda\in\Lambda_f\cap(-r_p,R_p)} \frac{1}{\lambda}$ and the following representation:

$$f(z) = e^{-az} \prod_{\lambda\in\Lambda_f} \left(1 - \frac{z}{\lambda}\right) e^{\frac{z}{\lambda}} \ . \tag{3.5.1}$$

So, $f \in \mathcal{LP}^0_{II}$. Rename zeros of f by: $\{\lambda_k\}_{k\geq 0} \cup \{-\lambda_{-l}\}_{l\geq 1} := \Lambda_f$, $0 < \lambda_k \leq \lambda_{k+1}$, $0 < \lambda_{-k-1} \leq \lambda_{-k-2}$ $\forall\, k \in \mathbb{Z}_0$, and for arbitrary $n \geq 0$, $m \geq 1$ set:

$$S(n,m) := \lim_{p\to\infty} \left(\sum_{k=n+1}^{n_+(p)} \frac{1}{\lambda_k} \; - \; \sum_{l=m+1}^{n_-(p)} \frac{1}{\lambda_{-l}} \right) \; = \; a - \sum_{k=0}^{n} \frac{1}{\lambda_k} \; + \; \sum_{l=1}^{m} \frac{1}{\lambda_{-l}} \ , \tag{3.5.2}$$

where $n_+(p) := \max\{k \in \mathbb{Z}_0 \mid \lambda_k < R_p\}$, $n_-(p) := \max\{l \geq 1 \mid \lambda_{-l} < r_p\}$, $p \geq 1$. Then for

$$\mathcal{R}_{n,m}(z) := \lim_{p\to\infty} \left[\prod_{k=n+1}^{n_+(p)} \left(1 - \frac{z}{\lambda_k}\right) \prod_{l=m+1}^{n_-(p)} \left(1 + \frac{z}{\lambda_{-l}}\right) \right] \ ,$$

using inequality $\log(1+x) \leq x \ \ \forall\, x > -1$, we get the following estimate:

$$0 < \mathcal{R}_{n,m}(x) \leq e^{-xS(n,m)} \ \ \forall\, x \in [-\lambda_{-m}, \lambda_n], \ \ n \geq 0, \ \ m \geq 1 \ . \tag{3.5.3}$$

For fixed arbitrary $N \geq 1$ let us find now such positive integers p_N, q_N that polynomial

$$P_N(z) := \prod_{k=0}^{p_N} \left(1 - \frac{z}{\lambda_k}\right) \prod_{l=1}^{q_N} \left(1 + \frac{z}{\lambda_{-l}}\right)$$

will satisfy conditions of the theorem.

Choosing subsequences of $\{R_N\}_{N\geq 1}$ and $\{r_N\}_{N\geq 1}$ and reindexed them we can consider that $R_N > \lambda_{n_f^+(N)+1}$, $r_N > \lambda_{n_f^-(N)+1}$ $\forall\ N \geq 1$, where $n_f^+(N) := \max\{k \in \mathbb{Z}_0 \mid$

$\lambda_k < N \}$, $n_f^-(N) := \max\{l \geq 1 \mid \lambda_{-l} < N\}$. Therefore inequalities $p_N \geq n_+(N)$, $q_N \geq n_-(N)$ yield validity of (3.3.11a).

Denote $S(q) := S(n_+(q), n_-(q))$, $q \geq 1$, and observe that $\lim\limits_{q\to\infty} S(q) = 0$.

Let $S(N) = 0$. Setting $p_N := n_+(N)$, $q_N := n_-(N)$ we get from (3.5.3) validity of (3.3.11b).

Let $S(N) > 0$. Since function $\varphi_+(n) := S(n_+(N)+n, n_-(N))$, $n \in \mathbb{Z}_0$, decreases from $S(N)$ to $-\infty \leq -\sum\limits_{l=1+n_-(N)}^{\infty} \frac{1}{\lambda_{-l}} < 0$, then it is possible to find such $r \in \mathbb{Z}_0$ that $\varphi_+(1+r) \leq 0 < \varphi_+(r)$, where $\varphi_+(1+r) = \varphi_+(r) - 1/\lambda_{1+r+n_+(N)}$. It follows from (3.5.3) that

$$\mathcal{R}_{n_+(N)+r+1, n_-(N)}(x) \leq \begin{cases} 1, & \forall\, x \in [-\lambda_{-n_-(N)}, 0] \ ; \\ \exp\left(-x\varphi_+(r) + \frac{x}{\lambda_{1+r+n_+(N)}}\right) \ \leq \ e \ , & \forall\, x \in [0, \lambda_{1+r+n_+(N)}] \ , \end{cases}$$

i.e. (3.3.11b) will be true for $p_N := 1 + r + n_+(N)$ and $q_N := n_-(N)$.

Let $S(N) < 0$. Since the function $\varphi_-(m) := S(n_+(N), n_-(N) + m)$, $m \geq 0$, increases from $S(N) < 0$ to $0 < \sum\limits_{k\geq 1+n_+(N)} \frac{1}{\lambda_k} \leq +\infty$, then one can find such $r \in \mathbb{Z}_0$ that $\varphi_-(r) < 0 \leq \varphi_-(r+1)$ and, obviuosly, $\varphi_-(r+1) = \varphi_-(r) + 1/\lambda_{-(r+1+n_-(N))}$. As well as in the previous case using inequality (3.5.3) we obtain validity of (3.3.11b) for $p_N := n_+(N)$, $q_N := 1 + r + n_-(N)$.

Observe now that according to our choice $\lim\limits_{N\to\infty} S(p_N, q_N) = 0$ and therefore by (3.5.2) we get:

$$\sum_{k=0}^{p_N} \frac{1}{\lambda_k} \ - \ \sum_{l=1}^{q_N} \frac{1}{\lambda_{-l}} \ = \ a - S(p_N, q_N) \to a \ , \quad N \to \infty \ ,$$

i.e. due to (3.5.1) for arbitrary $z \in \mathbb{C}$:

$$P_N(z) := e^{-(a - S(p_N,q_N))z} \cdot \prod_{k=0}^{p_N} \left(1 - \frac{z}{\lambda_k}\right) e^{\frac{z}{\lambda_k}} \cdot \prod_{l=1}^{q_N} \left(1 + \frac{z}{\lambda_{-l}}\right) e^{\frac{z}{\lambda_{-l}}} \ \to \ f(z) \ , \quad N \to \infty \ ,$$

and moreover, that convergence is uniform on any compact subset of $\mathbb{C}$. Theorem 3.2 is proved.

CHAPTER IV. Criterion of the polynomial density in $L_p(\mu)$

4.1. Representation Theorem. Let $w(x)$ be a positive and continuous function of real x such that for each $n = 0, 1, 2, \ldots,$ $x^n w(x)$ is bounded on the whole real line. In 1924 S. Bernstein [10] asked for conditions on w that algebraic polynomials $\mathcal{P}$ are dense in the space C_w^0 . In 1959 L.de Branges [12] gived the following its solution.

De Branges Theorem.([12]) *If $w : \mathbb{R} \to (0, +\infty)$, $w \in C(\mathbb{R})$ and $\|x^n w\|_{C(\mathbb{R})} < \infty$ $\forall\ n \in \mathbb{Z}_0$, then $\mathrm{Close}_{C_w^0}\mathcal{P} \neq C_w^0$ if and only if there exists a real entire function F of exponential type all whose zeros Λ_F are real and simple and which satisfies:*

$$\int_{\mathbb{R}} \frac{\log^+ |F(x)|}{1+x^2}\, dx \ < \ +\infty\ , \quad \sum_{\lambda \in \Lambda_F} \frac{1}{w(\lambda)|F'(\lambda)|} \ < \ \infty.$$

It should be noted that conditions on w in that theorem in view of Corollary 1.1 mean that C_w^0 is a Banach space, and by Theorem 3.1 and Corollary 3.1 conditions on F signify in fact that $F \in \mathcal{H}$.

In 1996 M. Sodin and P. Yuditskii [41] found a simpler proof of de Branges theorem and gived its version with weakened conditions on w .

Sodin-Yuditskii Theorem.([41]) *Let $w : \mathbb{R} \to [0, 1]$ is an upper semicontinuous function on $\mathbb{R}$ and $\|x^n w\|_{C(\mathbb{R})} < \infty$ $\forall\ n \in \mathbb{Z}_0$. Algebraic polynomials $\mathcal{P}$ are not dense in C_w^0 if and only if there exists such $B \in \mathcal{H} \cap \mathcal{E}_0$ with zeros $\Lambda_B \subseteq S_w := \{x \in \mathbb{R} \mid w(x) > 0\ \}$ that*

$$\sum_{\lambda \in \Lambda_B} \frac{1}{w(\lambda)|B'(\lambda)|} \ < \ \infty. \tag{4.1.1}$$

Observe, that if (4.1.1) is valid for some $B \in \mathcal{H}$ then according to the established by Hamburger [16, 2] property of such functions: $\sum_{\lambda \in \Lambda_B} \frac{\lambda^n}{B'(\lambda)} = 0$ $\forall\ n \in \mathbb{Z}_0$, we obtain that defined by equality $d\mu := \sum_{\lambda \in \Lambda_B} \frac{\delta_\lambda}{B'(\lambda)}$ measure μ , where δ_λ denotes Dirac's measure at the point λ , belongs to $\mathcal{M}(\mathbb{R})$ and evaluates (see Th. 1.3) on C_w^0 linear continuous functional $\int_{\mathbb{R}} f(x) d\mu(x)$ vanishing at all exponential functions. That is why in that case $\mathcal{P}$ is not dense in C_w^0 and taking into account indicated in Chapter I coincidence of the seminormed spaces C_h^0 and $C_{M_h}^0$ for arbitrary $h : \mathbb{R} \to [0, 1]$, we can reformulate aforementioned theorems as follows.

Proposition 4.1.([12, 41]) *Let $h : \mathbb{R} \to [0, 1]$, $\|x^n h\|_{C(\mathbb{R})} < \infty$ $\forall\ n \in \mathbb{Z}_0$, M_h is an upper Bair function of h and $S_{M_h} := \{x \in \mathbb{R} \mid M_h(x) > 0\ \}$. Then the following statements hold:*

(1) *if algebraic polynomials* $\mathcal{P}$ *are not dense in* C_h^0 *then there exists such* $B \in \mathcal{H} \cap \mathcal{E}_0$ *that*

$$\Lambda_B \subseteq S_{M_h} \text{ and } \sum_{\lambda \in \Lambda_B} \frac{1}{M_h(\lambda)|B'(\lambda)|} < \infty \tag{4.1.2}$$

(2) *if there exists satisfying (4.1.2)* $B \in \mathcal{H}$ *, then algebraic polynomials* $\mathcal{P}$ *are not dense in* C_h^0 *.*

For any positive integer N let $\mathcal{P}_N^*$ denote the set of real algebraic polynomials P of degree N with real and simple zeros only and $P(0) = 1$. Note that the proof of Theorem 3.3 shows that the closure of intersection (3.4.5) and $\mathcal{P}_N^*$ is a compact set in the topology $\tau_{\mathcal{E}}$. Using the Theorems 3.2, 3.3 and Corollary 3.2 it is easy to get the validity of the following assertion.

Lemma 4.1. *Let* $w : \mathbb{R} \to [0,1]$ *,* $\|x^n w\|_{C(\mathbb{R})} < \infty \ \ \forall n \in \mathbb{Z}_0$ *,* w *is an upper semicontinuous function on* $\mathbb{R}$ *,* $\sigma := \chi_{S_w}(0) \ \in \{0,1\}$ *and function* $\theta : [0,+\infty) \to \mathbb{R}$ *for some finite constants* $C, c, \alpha > 0$ *satisfies* $ce^{-\alpha x} \le \theta(x) \le \frac{C}{1+x} \ \ \forall x \ge 0$ *. Algebraic polynomials* $\mathcal{P}$ *are dense in* C_w^0 *if and only if :*

$$\lim_{N \to \infty} \ \min_{P \in \mathcal{P}_N^*} \left(\sum_{\lambda \in \Lambda_P} \frac{\theta(|\lambda|)}{|\lambda|} + \sum_{\lambda \in \Lambda_P} \frac{1}{w(\lambda)|\lambda|^{\sigma}|P'(\lambda)|} \right) = +\infty \tag{4.1.3}$$

Calling to mind (see 1.1) that $\mathcal{B}(\mathbb{R})$ denotes the family of Borel subsets of $\mathbb{R}$ we formulate now (see also Proposition 2.2) the main result of this paper.

Theorem 4.1. *Let* $1 \le p < \infty$ *and* μ *be a positive Borel measure on* $\mathbb{R}$ *with all finite moments:* $\int_{\mathbb{R}} |x|^n \ d\mu(x) < \infty \ \ \forall \ n \in \mathbb{Z}_0$ *, and unbounded support:* $\operatorname{supp}\mu := \{x \in \mathbb{R} \mid \mu(x-\delta, x+\delta) > 0 \ \ \forall \ \delta > 0\}$ *.*

Algebraic polynomials $\mathcal{P}$ *are dense in the space* $L_p(\mathbb{R}, d\mu)$ *if and only if the measure* μ *can be represented in the following form:*

$$\mu(A) = \int\limits_A w(x)^p \ d\nu(x) \quad \forall \ A \in \mathcal{B}(\mathbb{R}) \ , \tag{4.1.4}$$

where ν *is some finite positive Borel measure on* $\mathbb{R}$ *and* w *is some upper semicontinuous on* $\mathbb{R}$ *function* $w : \mathbb{R} \to [0,1]$ *,* $\|x^n w\|_{C(\mathbb{R})} < \infty \ \ \forall \ n \in \mathbb{Z}_0$ *, for which algebraic polynomials* $\mathcal{P}$ *are dense in the seminormed space* C_w^0 *, i.e.* w *satisfies (4.1.3).*

It is interesting to note that by Theorem 1.3 representation (4.1.4) for the measure μ holds if and only if $L(f) := \int_{\mathbb{R}} f(x) \ d\mu(x)$ is a linear continuous functional on the seminormed space $C_{w^p}^0$. Another equivalent to Theorem 4.1 statements and their important corollaries will be given in the second part of that paper.

4.2. Preliminary Lemmas.

4.2.1. Formulations.

For arbitrary function $f : \mathbb{R} \to [-\infty, +\infty]$, denote

$$\mathrm{dom} f := \{x \in \mathbb{R} \mid f(x) < +\infty \} \ , \quad \mathrm{epi} f := \left\{(x, y) \in \mathbb{R}^2 \mid y \geq f(x) \right\} \ . \tag{4.2.1}$$

Lemma 4.2. *Let μ be a positive Borel measure on $\mathbb{R}$ and function $\alpha : \mathbb{R} \to [0, +\infty]$ is lower semicontinuous on $\mathbb{R}$ with $\mu(\mathrm{dom}\alpha) > 0$. Then there exists such lower semicontinuous on $\mathbb{R}$ function β that:*

(4.2.2a) $\beta(x) \geq \alpha(x) \ \ \forall \, x \in \mathbb{R}$; (4.2.2b) $\mu \left(x \in \mathbb{R} \mid \beta(x) \neq \alpha(x) \ \right) = 0$;

(4.2.2c) $\mu \left(y \in \mathbb{R} \mid |x - y| + |\beta(x) - \beta(y)| < \varepsilon \right) > 0 \quad \forall \, \varepsilon > 0 \quad \forall \, x \in \mathrm{dom}\beta$.

Denote by $\mathcal{K}_{\mathcal{R}}$ the class of entire functions f satisfying conditions:

(3.3.14a) f is a real function with only real and simple zeros Λ_f ;

(3.3.14b) if $\mathrm{co}\Lambda_f = \mathbb{R}$ then f is of exponential type, but if $\mathrm{co}\Lambda_f \neq \mathbb{R}$ then f is of minimal exponential type ;

(3.3.17) $\displaystyle\limsup_{|\lambda|\to\infty, \ \lambda\in\Lambda_f} \frac{\log^+ \frac{1}{|f'(\lambda)|}}{\log|\lambda|} < +\infty$.

As it was noted after the Corollary 3.1 every function $f \in \mathcal{K}_{\mathcal{R}}$ can be restored by its zeros up to a constant factor:

$$f(z) = f^{(q)}(0) \cdot z^q \cdot \lim_{R\to+\infty} \prod_{|\lambda|<R, \ \lambda\in\Lambda_f\setminus\{0\}} \left(1 - \frac{z}{\lambda}\right) \ , \quad q \in \{0, 1\} \ , \quad z \in \mathbb{C} \ . \tag{4.2.3}$$

Lemma 4.3. *Let $X \in \{\mathcal{K}_{\mathcal{R}}, \mathcal{K}, \mathcal{H}\}$. For arbitrary $B \in X$ with zeros $\{a_n\}_{n\geq1} := \Lambda_B$ there exist such constant $C > 0$ and such sequence of real positive numbers $\{\delta_n\}_{n\geq1}$ that for any sequence of real numbers $\{b_n\}_{n\geq1}$ satisfying condition:*

$$|b_n - a_n| \leq \delta_n \ \ \forall \, n \geq 1 \tag{4.2.4}$$

it is possible to find such $D \in X$ that $\Lambda_D = \{b_n\}_{n\geq1}$ and

$$|B'(a_n)| \leq C \cdot |D'(b_n)| \quad \forall \, n \geq 1 \ . \tag{4.2.5}$$

4.2.2. Proof of Lemma 4.2.

For arbitrary $A \subseteq \mathbb{R}^2$ denote $P[A] := \{x \in \mathbb{R} \mid \exists y \in \mathbb{R} : (x, y) \in A\}$ and let E_α denote the set of those points $\overline{x} := (x, y) \in \mathrm{epi}\alpha$, for which one can find such $\varepsilon(\overline{x}) > 0$ that:

$$\mu \left(P \left[\ (\ \overline{x} + \varepsilon(\overline{x})V \ ) \ \cap \ \mathrm{epi}\alpha \ \right]\right) \ = \ 0, \tag{4.2.6}$$

where $V := \{(x,y) \in \mathbb{R}^2 \mid |x| + |y| < 1\}$. The known property of separable metric spaces [3, I.5, Lemma 2] means that from the open covering of the set

$$G := \bigcup_{\overline{x} \in E_\alpha} (\overline{x} + \varepsilon(\overline{x})V) \tag{4.2.7}$$

it is possible to extract the countable subcovering: $G = \bigcup_{n \geq 1} G_n$, $G_n := \overline{x}_n + \varepsilon(\overline{x}_n)V$, $\overline{x}_n \in E_\alpha \quad \forall\, n \geq 1$. That is why equality (4.2.6) and countable semiadditivity of the measure μ imply : $\mu(P[G \cap \mathrm{epi}\alpha]) = \mu\left(\bigcup_{n \geq 1} P[G_n \cap \mathrm{epi}\alpha]\right) \leq 0$, i.e.

$$\mu(P[\ G\ \cap\ \mathrm{epi}\alpha\ ]) = 0\ . \tag{4.2.8}$$

Since every point in $G \cap \mathrm{epi}\alpha$ possesses the property (4.2.6) then the set $B := (\mathrm{epi}\alpha) \setminus G$ contains all those points $\overline{x} \in \mathrm{epi}\alpha$ for which

$$\mu(P[\ (\ \overline{x} + \varepsilon V\ )\ \cap\ \mathrm{epi}\alpha\ ]) > 0 \quad \forall\, \varepsilon\ > 0\ . \tag{4.2.9}$$

In addition an evident inclusion $P[(\overline{x} + \varepsilon V) \cap \mathrm{epi}\alpha] \subseteq P[(\overline{x} + (0,t) + \varepsilon V) \cap \mathrm{epi}\alpha] \ \ \forall\, t \geq 0$, $\varepsilon > 0$, $\overline{x} \in \mathbb{R}^2$, yields $B + (0,t) \subseteq B\ \forall\, t \geq 0$. Therefore the set B coincides with epigraph $\mathrm{epi}\beta$ of the lower semicontinuous on $\mathbb{R}$ function β defined by formula:

$$\beta(x) := \begin{cases} +\infty\ , & x \notin P[B]\ ; \\ \min\{y \in \mathbb{R} \mid (x,y) \in B\}\ , & x \in P[B]\ . \end{cases}$$

Validity of (4.2.2a) follows from $\mathrm{epi}\beta \subseteq \mathrm{epi}\alpha$. Due to (4.2.8): $\mu(P[\mathrm{epi}\alpha\ \setminus\ \mathrm{epi}\beta]) = \mu(P[G \cap \mathrm{epi}\alpha]) = 0$, and so (4.2.2b) is also true. Since by (4.2.9) for any $\varepsilon > 0$ and $\overline{x} \in \mathrm{epi}\beta$:

$$0 < \mu(P[(\overline{x} + \varepsilon V) \cap \mathrm{epi}\alpha]) = \mu\left(P[(\overline{x} + \varepsilon V) \cap \mathrm{epi}\beta] \cup P[(\overline{x} + \varepsilon V) \cap (\mathrm{epi}\alpha \setminus \mathrm{epi}\beta)]\right) \leq$$

$$\leq \mu(P[(\overline{x} + \varepsilon V) \cap \mathrm{epi}\beta])\ ,$$

then the property (4.2.2c) is fulfilled. That is why β satisfies all required conditions and Lemma 4.2 is proved.

4.2.3. Proof of Lemma 4.3. Since the Lemma's statement is invariant with respect to the substitution x by $x + a$, $a \in \mathbb{R}$, we can without loss of generality consider $0 \notin \Lambda_B$. Assuming $0 < |a_1| \leq |a_2| \leq \ldots \leq |a_n| \leq \ldots$, let us set

$$\rho_k := \min\{1, |a_k|, \{\ |\ |a_k| - |a_m|\ |\ \ \mid m \geq 1,\ |a_m| \neq |a_k|\}\ \},\ k \geq 1\ . \tag{4.2.10}$$

An elementary reasonings show that for real constants $\alpha, \beta, \rho, \Delta$, satisfying

$$\alpha \in \mathbb{R} \setminus \{0\}\ ;\ \ 0 < \rho < |\alpha|,\ \ 0 < \Delta \leq \frac{1}{2}\rho\ ,\ \ |\alpha - \beta| \leq \Delta\ , \tag{4.2.11}$$

the following inequality holds:

$$\left| \left(1 - \frac{x}{\alpha}\right) \cdot \left(1 - \frac{x}{\beta}\right)^{-1} \right| \leq\ 1 + 4\frac{\Delta}{\rho} \quad \forall x \in \mathbb{R}:\ |x - \alpha| \geq \rho\ . \tag{4.2.12}$$

As for every $k \geq 1$ function $B_k(x) := (1 - \frac{x}{a_k})^{-1} B(x)$ is continuous on $\mathbb{R}$ and $B'(a_k) = -a_k B'(a_k)$, then there exists such $\alpha_k > 0$ that

$$|B_k(x)| \geq \frac{1}{2}\ |a_k|\ |B'(a_k)| \quad \forall\ x \in \mathbb{R} :\ |x - a_k| \leq \alpha_k\ . \tag{4.2.13}$$

Let us set

$$\delta_k := \min\left\{\alpha_k, \frac{\rho_k}{4(1+a_k^2)}\right\} \quad \forall\ k \geq 1\ , \tag{4.2.14}$$

and consider an arbitrary sequence $\{b_k\}_{k\geq 1}$ satisfying inequalities (4.2.4). Since $\sum_{k\geq 1} \frac{1}{|a_k|^{1+\varepsilon}} < \infty \ \ \forall \varepsilon > 0$ and $\left|\frac{1}{b_k} - \frac{1}{a_k}\right| \leq \frac{2}{a_k^2} \ \ \forall\ k \geq 1$, then due to (4.2.3) one can determine an entire function of exponential type by the folowing equality:

$$D(z) := \lim_{R\to+\infty} \prod_{|b_k|<R,\ k\geq 1} \left(1 - \frac{z}{b_k}\right)\ , \tag{4.2.15}$$

which, obviously, possesses properties (3.3.14a) and (3.3.14b). It is easy to verify that (4.2.10) and (4.2.14) give possibility to choose such tending to infinity sequence of positive real numbers R_p , $p \geq 1$, that interval $(-R_p, R_p)$ for every $p \geq 1$ will include the same number N_p of zeros of the functions $D(z)$ and $B(z)$. Therefore the following relation holds:

$$\frac{B_k(b_k)}{(-b_k)D'(b_k)} = \lim_{p\to\infty} \prod_{m=1,\ m\neq k}^{N_p} \left(1 - \frac{b_k}{b_m}\right)^{-1} \left(1 - \frac{b_k}{a_m}\right)\ , \quad k \geq 1\ . \tag{4.2.16}$$

Applying estimate (4.2.12) we get: $\left|\left(1 - \frac{b_k}{b_m}\right)^{-1} \left(1 - \frac{b_k}{a_m}\right)\right| \leq 1 + 4\frac{\delta_m}{\rho_m} \leq 1 + \frac{1}{1+a_m^2}$, $k, m \geq 1$, $m \neq k$, and so by (4.2.13) and (4.2.14):

$$C := 8 \cdot \exp\left\{\sum_{m\geq 1} \frac{1}{1+a_k^2}\right\} \geq 4\frac{|a_k|}{|b_k|}\frac{|B'(b_k)|}{|D'(b_k)|} \geq \frac{|B'(a_k)|}{|D'(b_k)|} \quad \forall\ k \geq 1\ . \tag{4.2.17}$$

Since defined in (4.2.17) constant C does not depend on choice of the sequence $\{b_k\}_{k\geq 1}$ then (4.2.17) represents the required inequality (4.2.5), from where by the Theorem 3.1 and Corollary 3.1 we will have $D \in X$ for any indicated in Lemma 4.3 choice of the class X . Lemma 4.3 is proved.

4.3. Proof of Theorem 4.1.

Sufficiency. Since $\frac{1}{w} \in L_p(\mu)$ then the density in $L_p(\mu)$ of all compactly supported continuous on $\mathbb{R}$ functions and an evident inequality $\|f\|_{L_p(\mu)} \leq \|f\|_w \cdot \left\|\frac{1}{w}\right\|_{L_p(\mu)} \ \forall\ f \in C_w^0$ by virtue of the Proposition 2.2 means the density $\mathcal{P}$ in $L_p(\mu)$.

Necessity.

4.3.1^0. By (2.2.1c) density of $\mathcal{P}$ in $L_p(\mu)$ is equivalent to the existence of such sequence of polynomials $P_n \in \mathcal{P}[\mathbb{C}]$, $n \geq 1$, that

$$\alpha_n := \left\|\frac{1}{x+i} - P_n\right\|_{L_p(\mu)}^p \ \to\ 0\ , \quad n \to \infty\ , \tag{4.3.1}$$

where without loss of generality we can assume that $\sum_{n\geq 1}\alpha_n \leq 1$. Then nondecreasing sequence of nonnegative continuous on $\mathbb{R}$ functions

$$\varphi_N(x) := \sum_{n=1}^{N}\left|\frac{1}{x+i} - P_n(x)\right|^p \ , \quad N \geq 1 \tag{4.3.2}$$

satisfies $\|\varphi_N\|_{L_1(\mu)} \leq 1 \ \ \forall\ N \geq 1$ and by Beppo-Levi theorem has a limit $\varphi \in L_1(\mu)$: $\|\varphi\|_{L_1(\mu)} \leq 1$. It is easy to see also that φ is a lower semicontinuous function and $\mu(\mathrm{dom}\varphi) > 0$.

4.3.2^0. Under conditions of the Theorem 4.1: $0 < s_n := \int_{\mathbb{R}} |x|^n \ d\mu(x) < \infty \ \ \forall\ n \in \mathbb{Z}_0$. Therefore the function

$$h(x) := 2s_0 \cdot \sum_{n\geq 0} \frac{1}{2^{n+1}} \frac{|x|^n}{s_n} \ , \quad x \in \mathbb{R} \ ,$$

has the following properties: $h \in C(\mathbb{R})$, $h(x) \in [1, +\infty) \ \ \forall\ x \in \mathbb{R}$, $2s_0 = \int_{\mathbb{R}} h(x) \ d\mu(x)$ and $\inf_{x\in\mathbb{R}}(1+x^{2n})^{-1} \cdot h(x) > 0 \ \forall\ n \in \mathbb{Z}_0$. Let

$$a(x) := h(x) + \varphi(x) \ , \quad x \in \mathbb{R} \ . \tag{4.3.3}$$

Then for $f = a$ the following conditions hold:

(4.3.4a) $f(x) \geq 1 \ \ \forall\ x \in \mathbb{R}$; (4.3.4b) $\inf\limits_{x\in\mathbb{R}}(1+x^{2n})^{-1} \cdot f(x) > 0 \ \forall\ n \in \mathbb{Z}_0$;
(4.3.4c) f is lower semicontinuous on $\mathbb{R}$; (4.3.4d) $f \in L_1(\mu)$.

4.3.3^0. By (4.3.4a) and (4.3.4d) with $f = a$, the sequence of positive numbers

$$t_n := \int\limits_{|x|>n} a(x) \ d\mu(x) \ , \quad n \in \mathbb{Z}_0 \ ,$$

tends to zero as $n \to \infty$, and therefore one can find such subsequence $\{n_k\}_{k\in\mathbb{Z}_0}$, $n_0 := 0$, that $\sum_{k\geq 0}\sqrt{t_{n_k}} < \infty$ and $t_{n_{k+1}} < t_{n_k} \ \ \forall\ k \in \mathbb{Z}_0$. Then for the function

$$\frac{1}{\theta(x)} := \sqrt{t_0}\left(\frac{\chi_{\{0\}}(x)}{\sqrt{t_0}} + \sum_{k\geq 0}\frac{\chi_{(n_k,n_{k+1}]}(|x|)}{\sqrt{t_{n_k}}}\right)$$

we have $\theta(x) \to 0$, $|x| \to +\infty$, $\theta(x)$ is an even lower semicontinuous on $\mathbb{R}$ function, which does not increase as $x \geq 0$, $\theta(x) \in (0,1] \ \ \forall\ x \in \mathbb{R}$ and

$$\int\limits_{\mathbb{R}} \frac{a(x)}{\theta(x)} \ d\mu(x) = a(0) \cdot \mu(\{0\}) + \sum_{k\geq 0}\frac{t_{n_k} - t_{n_{k+1}}}{\sqrt{t_{n_k}}} \ < \ \infty \ .$$

That is why all properties (4.3.4a-d) are valid and for $f = \alpha_0$, where $\alpha_0(x) := \frac{a(x)}{\theta(x)}$, $x \in \mathbb{R}$. Applying Lemma 4.2 to the function α_0 we obtain the function α for which all conditions (4.3.4a-d) with $f = \alpha$ will be true and also:

(4.3.4e) $\alpha(x) \geq \frac{a(x)}{\theta(x)} \quad \forall\ x \in \mathbb{R}$;

(4.3.4g) $\mu\,(\, y \in \mathbb{R} \mid |x-y| + |\alpha(x) - \alpha(y)| < \varepsilon) > 0 \ \ \forall\, \varepsilon > 0 \ \ \forall\, x \in \mathrm{dom}\alpha$.

4.3.4^0. In view of (4.3.1) and (4.3.4a) with $f = \alpha$ we can apply known Riesz's theorem to the convergent to zero in the space $L_1(\mu)$ sequence $\frac{1}{\alpha(x)} \left|\frac{1}{x+i} - P_n(x)\right|^p$, $n \geq 1$ (we consider here $1/+\infty := 0$). That is why taking into account $\mu(\mathbb{R} \setminus \mathrm{dom}\alpha) = 0$ we can find such $A \subseteq \mathbb{R}$, $\mu(A) = 0$, and such subsequence $\{n_k\}_{k \geq 1}$ that

$$\lim_{k \to \infty} \frac{1}{\alpha(x)} \left|\frac{1}{x+i} - P_{n_k}(x)\right|^p = 0 \ \ \forall\, x \in \mathbb{R} \setminus A \ ; \ \ \mathbb{R} \setminus A \subseteq \mathrm{dom}\alpha \ . \tag{4.3.5}$$

On the other hand for arbitrary $T > 0$, $x \in \mathrm{dom}\alpha$ and $k \geq 1$ properties (4.3.4e), (4.3.3) and (4.3.2) yield:

$$\frac{1}{\alpha(x)} \left|\frac{1}{x+i} - P_{n_k}(x)\right|^p \leq \frac{\left|\frac{1}{x+i} - P_{n_k}(x)\right|^p}{a(x)} \theta(x) \leq \theta(T) \quad \forall\, |x| \geq T \ ,$$

from where

$$\lim_{T \to +\infty} \ \sup_{|x| \geq T} \frac{1}{\alpha(x)} \left|\frac{1}{x+i} - P_{n_k}(x)\right|^p = 0 \ . \tag{4.3.6}$$

By virtue of the Proposition 2.2 established properties (4.3.5), (4.3.6) mean that for any countable set $\Lambda \subseteq \mathbb{R} \setminus A$, which has not finite limit points the following statement holds:

$$\mathcal{P} \ \ \textit{is dense in} \ \ \ C^0_\beta \ \ , \ \ \textit{where} \ \beta(x) := \frac{\chi_\Lambda(x)}{\alpha(x)^{1/p}} \ , \ \ \ x \in \mathbb{R} \ . \tag{4.3.7}$$

4.3.5^0. Let us exhibit that in fact more strong than (4.3.7) statement is valid:

$$\mathcal{P} \ \ \textit{is dense in} \ \ \ C^0_w \ \ , \ \ \textit{where} \ w(x) := \frac{\chi_{\mathrm{supp}\mu}(x)}{\alpha(x)^{1/p}} \ , \ \ \ x \in \mathbb{R} \ . \tag{4.3.8}$$

Assuming a contrary we by Proposition 4.1 can find such $B \in \mathcal{H} \cap \mathcal{E}_0$, that $\Lambda_B \subseteq \mathrm{supp}\mu \cap \mathrm{dom}\alpha$ and

$$\sum_{\lambda \in \Lambda_B} \frac{\alpha(\lambda)^{1/p}}{|B'(\lambda)|} \ < \ \infty \ . \tag{4.3.9}$$

Applying Lemma 4.3 to the function $B \in \mathcal{H}$, we can find also corresponding to that function constant $C > 0$ and the sequence of positive real numbers $\{\delta_\lambda\}_{\lambda \in \Lambda_B}$. Determine now the numbers b_λ , $\lambda \in \Lambda_B$, satisfying $|b_\lambda - \lambda| \leq \delta_\lambda \ \ \forall\, \lambda \in \Lambda_B$.

If $\mu(\{\lambda\}) > 0$, then $\lambda \notin A$ and let $b_\lambda = \lambda$ in that case. If $\mu(\{\lambda\}) = 0$ then choose an arbitrary b_λ from the nonempty by (4.3.4g) set:

$$\{y \in \mathbb{R} \mid y \neq \lambda, \ |y - \lambda| + |\alpha(y) - \alpha(\lambda)| \leq \delta^*_\lambda \ \} \setminus A \ , \tag{4.3.10}$$

where $\delta^*_\lambda := \min\{\alpha(\lambda), \delta_\lambda\}$.

Then $\alpha(b_\lambda) \leq 2\alpha(\lambda) \ \ \forall\, \lambda \in \Lambda_B$ and constructed by such sequence $\{b_\lambda\}_{\lambda \in \Lambda_B}$ entire function $D \in \mathcal{H}$ in Lemma 4.3 will satisfy inequality: $|B'(\lambda)| \leq C|D'(b_\lambda)| \ \ \forall\, \lambda \in \Lambda_B$. In view of (4.3.9) this means that

$$\sum_{\lambda \in \Lambda_B} \frac{\alpha(b_\lambda)^{1/p}}{|D'(b_\lambda)|} \ < \ \infty \ . \tag{4.3.11}$$

Since the sequence $\{b_\lambda\}_{\lambda\in\Lambda_B} \subseteq \mathbb{R}\setminus A$ and has not the finite limit points then obtained inequality (4.3.11) contradicts (4.3.7) with $\Lambda = \{b_\lambda\}_{\lambda\in\Lambda_B}$. Thus, statement (4.3.8) has been proved.

It remains to observe that defined in (4.3.8) function w in view of (4.3.4a-d) with $f = \alpha$ is upper semicontinuous on $\mathbb{R}$ and satisfies : $\|x^n w\|_{C(\mathbb{R})} < \infty \ \ \forall\, n \in \mathbb{Z}_0$, $0 \le w(x) \le 1 \ \ \forall\, x \in \mathbb{R}$, $\frac{1}{w} \in L_p(\mu)$. That is why defined by the following equality

$$\nu(A) := \int\limits_A \frac{1}{w(x)^p}\, d\mu(x) \quad \forall\, A \in \mathcal{B}(\mathbb{R}) ,$$

measure ν will be finite positive Borel measure on the real axis. Since the bounded function $w(x)^p$ is Borel we by known change of variables theorem in the Lebesgue integral get the required representation of the measure μ : $\mu(A) = \int\limits_A w(x)^p\, d\nu(x) \ \ \forall\, A \in \mathcal{B}(\mathbb{R})$. Theorem 4.1 is proved.

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